

Host Algebras.

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Abstract *A host algebra generalises the concept of a group algebra in the following way. Take a unital C^* -algebra \mathcal{F} and a proper subset of its states \mathfrak{S}_0 within which one wants to keep the analysis (e.g. the group algebra of a discrete group G , and the set of its states continuous w.r.t. some nondiscrete group topology of G). Then a host algebra is a C^* -algebra \mathcal{L} for which we have embeddings $\mathcal{F} \subset \mathcal{E} \supset \mathcal{L}$ into a larger C^* -algebra \mathcal{E} , such that states on \mathcal{L} extend uniquely to \mathcal{F} , and this extension defines a norm continuous affine bijection between \mathfrak{S}_0 and the whole state space of \mathcal{L} . The main examples –though not the only ones– are of course group and covariance algebras. Here we study the general existence question for a host algebra of a given pair $(\mathcal{F}, \mathfrak{S}_0)$, we show that given a host algebra one can do integral decompositions of states in \mathfrak{S}_0 in terms of other states in \mathfrak{S}_0 , and we show that if one does induction of representations via host algebras, one stays within the class of representations with the right continuity properties w.r.t. \mathfrak{S}_0 . Moreover, up to a central algebra, one can always construct a host algebra (if \mathfrak{S}_0 is a folium), but this central algebra can be an obstruction to the existence of a host algebra. These results should be interesting to anyone who wants to construct a group algebra for general topological groups, and quantum physicists should also be interested due to various selection criteria for physically acceptable states.*

Keywords: host algebra, C^* -algebra, operator algebra, states, folium, group algebra, induction of representations, decomposition of states.

AMS classification: 46L05, 46L30, 81T05, 43A40

Introduction.

In quantum physics one is frequently given a unital C^* -algebra \mathcal{F} for the observables and a distinguished proper subset of states $\mathfrak{S}_0 \subset \mathfrak{S}(\mathcal{F})$ of its state space together with the constraint that the physical system can only realise these states and no others.

- Examples.** (1) Constrained systems;– here one has a distinguished set of unitaries $\mathcal{U} \subset \mathcal{F}_u$ and for the set of physically realisable states \mathfrak{S}_0 we have the Dirac states $\{\omega \in \mathfrak{S}(\mathcal{F}) \mid \omega(\mathcal{U}) = 1\}$, cf. Grundling and Hurst [GH].
- (2) The algebra of the canonical commutation relations $\mathcal{F} = \overline{\Delta(S, B)}$ over a symplectic space (S, B) (cf. Manuceau [Ma]), where \mathfrak{S}_0 is required to be the set of regular states.
- (3) In finite dimensional quantum mechanics, \mathcal{F} is taken as a factor of Type I and the physically relevant states, \mathfrak{S}_0 , as its set of normal states.
- (4) In algebraic quantum field theory, let \mathcal{F} be the inductive limit of a net of local algebras, and let \mathfrak{S}_0 be the set of locally normal states with respect to some distinguished state (cf. Haag [Ha]).

In such a situation one can object that the given system $(\mathcal{F}, \mathfrak{S}_0)$ is not satisfactory because it leads naturally to nonphysical objects, for instance the weak*-closure of \mathfrak{S}_0 can be the full state space $\mathfrak{S}(\mathcal{F})$. Indeed, from the point of view that the states and observables should be in some kind of duality (a Heisenberg–Schrödinger picture equivalence), one can argue that if \mathfrak{S}_0 is the physical state space, then \mathcal{F} is not the correct algebra of observables. From the mathematical point of view there are also problems, e.g. we may not have a decomposition theory of states in \mathfrak{S}_0 in terms of other states in \mathfrak{S}_0 . Another problem, is that if we have two pairs $(\mathcal{F}^{(i)}, \mathfrak{S}_0^{(i)})$, $i = 1, 2$ and an imprimitivity bimodule for the algebras, then when we induce a representation from one algebra to the other, we can easily move out of the class of allowed states. Our idea here is to replace \mathcal{F} by an algebra \mathcal{L} which has precisely \mathfrak{S}_0 as its state space, in a sense to be made precise below.

For some examples of $(\mathcal{F}, \mathfrak{S}_0)$, we do in fact have a more convenient algebra which in some sense has precisely \mathfrak{S}_0 as its state space, e.g. if we take $\mathcal{F} = \mathcal{B}(\mathcal{H})$, and let \mathfrak{S}_0 be its set of normal states, then the algebra of compact operators $\mathcal{L} = \mathcal{K}(\mathcal{H})$ is an algebra with state space \mathfrak{S}_0 in the sense that $\mathfrak{S}_0 \upharpoonright \mathcal{K}(\mathcal{H}) = \mathfrak{S}(\mathcal{K}(\mathcal{H}))$ and the states on $\mathcal{K}(\mathcal{H})$ extend uniquely. (We use here the notation $\mathfrak{S}(\cdot)$ for the state space of its argument). Group algebras provide another set of examples – see the next section. So, inspired by these examples, we will study here the following situation. Given a pair $(\mathcal{F}, \mathfrak{S}_0)$, find a pair of C^* -algebras $\mathcal{L} \subset \mathcal{E}$ and an embedding $\xi : \mathcal{F} \rightarrow \mathcal{E}$ such that the states on \mathcal{L} extend uniquely to $\xi(\mathcal{F})$, and $\theta(\mathfrak{S}(\mathcal{L})) = \mathfrak{S}_0$ where θ denotes the extension map $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}(\mathcal{F})$. Naturally, there are existence questions to be answered, and we will address these first. In the next section we develop our basic theory, in Sect. 2 we construct a host up to a central algebra, and in Sect. 3 we do a few applications.

1. Basic Concepts.

To prepare the ground, we first recall some background material.

Recall that a hereditary subalgebra \mathcal{B} of a C^* -algebra \mathcal{A} is a C^* -subalgebra such that $0 < A < B$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$ implies that $A \in \mathcal{B}$. Equivalently (cf. Murphy [Mu]) \mathcal{B} is hereditary if $\mathcal{B}\mathcal{A}\mathcal{B} \subseteq \mathcal{B}$. Such algebras are plentiful, and are in bijection with the set of closed left ideals of \mathcal{A} . All closed two-sided ideals are hereditary. For us, the most important property is: a C^* -subalgebra \mathcal{B} is hereditary iff each state $\omega \in \mathfrak{S}(\mathcal{B})$ has a unique extension to a state $\theta(\omega) \in \mathfrak{S}(\mathcal{A})$ (cf. Kusuda [Ku]). If we define a projection $P \in \mathcal{A}''$ as the unit of $\mathcal{B}'' \subset \mathcal{A}''$, then the map $\theta : \mathcal{B}^* \rightarrow \mathcal{A}^*$ defined by

$$\theta(\varphi)(A) := \varphi(PAP) = \lim_{\alpha} \varphi(E_{\alpha}AE_{\alpha}) \quad (1)$$

for all $A \in \mathcal{A}$ and some approximate identity $\{E_{\alpha}\}$ of \mathcal{B} , is precisely the unique extension map on the states $\mathfrak{S}(\mathcal{B}) \subset \mathcal{B}^*$. Moreover θ is an isometry, hence its range is norm closed. Given a representation of \mathcal{B} , we can always induce a representation on \mathcal{A} from it (cf. Fell and Doran, XI.7.6 [FD]), but usually this will be on a different space than the original representation. In the case that \mathcal{B} is a two-sided ideal of \mathcal{A} we have that $P \in \mathcal{A}' \cap \mathcal{A}''$, and so

$$\theta(\varphi)(A) := \varphi(PA) = \lim_{\alpha} \varphi(E_{\alpha}A) \quad (2)$$

and now, in addition, representations of \mathcal{B} also extend uniquely on the same space to \mathcal{A} . For a representation $\pi : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$, the unique extension is

$$\tilde{\pi}(A) := \pi(PA) = \text{s-lim}_{\alpha} \pi(E_{\alpha}A) \quad (3)$$

In this paper we will always use the notation $(\mathcal{F}, \mathfrak{S}_0)$, to denote a unital C^* -algebra \mathcal{F} and a distinguished subset of its state space. We choose \mathcal{F} to be unital, since this ensures that its state space is w^* -closed, hence norm closed (cf. Pedersen [Pe] 3.2.1). We are now ready for our basic definitions:

Def. Given a pair $(\mathcal{F}, \mathfrak{S}_0)$ consisting of a unital C^* -algebra and a proper subset of its states, we say a C^* -algebra \mathcal{L} is a *host* for the pair if there is a unital C^* -algebra $\mathcal{E} \supset \mathcal{L}$ (faithful embedding) and a unital $*$ -homomorphism $\xi : \mathcal{F} \rightarrow \mathcal{E}$ such that:

- (i) \mathcal{L} is hereditary in \mathcal{E} ,
- (ii) \mathcal{E} is generated by \mathcal{L} and $\xi(\mathcal{F})$,
- (iii) the map $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}(\mathcal{F})$ defined by unique extension

$$\theta(\omega)(F) = \lim_{\alpha} \omega(E_{\alpha}\xi(F)E_{\alpha}), \quad F \in \mathcal{F} \quad (4)$$

for any approximate identity $\{E_{\alpha}\}$ of \mathcal{L} , is injective and has range $\theta(\mathfrak{S}(\mathcal{L})) = \mathfrak{S}_0$.

- In the case that \mathcal{L} is in addition a two sided ideal of \mathcal{E} , we call it an *ideal host*.

Remarks (1) The requirement (ii) that \mathcal{E} be generated by \mathcal{L} and $\xi(\mathcal{F})$, is not essential;— if we start from some larger algebra \mathcal{E} satisfying the other requirements, we can

always replace \mathcal{E} by $C^*(\mathcal{L} \cup \xi(\mathcal{F}))$ inside \mathcal{E} because \mathcal{L} is hereditary for any subalgebra of \mathcal{E} containing it. An important point in the definition, is that we allow \mathcal{L} to be outside \mathcal{F} .

- (2) If $(\mathcal{F}, \mathfrak{S}_0)$ has an ideal host $\mathcal{L} \subset \mathcal{E}$, then there is a natural homomorphism of \mathcal{E} into the multiplier algebra $M(\mathcal{L}) \subset \mathcal{L}''$, and so we can equivalently define an ideal host for $(\mathcal{F}, \mathfrak{S}_0)$ as a C^* -algebra \mathcal{L} together with a homomorphism $\xi : \mathcal{F} \rightarrow M(\mathcal{L})$ such that $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}(\mathcal{F})$ is injective, with image \mathfrak{S}_0 .
- (3) If $(\mathcal{F}, \mathfrak{S}_0)$ has only a host $\mathcal{L} \subset \mathcal{E}$, then we may want to try a similar embedding than in the last remark. The point is that \mathcal{E} acts as a set of quasi-multipliers of \mathcal{L} (i.e. $\mathcal{L}E\mathcal{L} \subset \mathcal{L}$ for each $E \in \mathcal{E}$) and we know by C. Akemann and G. Pedersen, Prop. 4.2 [AP], that there is a linear bijection between the quasi-multipliers and elements $A \in \mathcal{L}''$ such that $\mathcal{L}A\mathcal{L} \subset \mathcal{L}$. However, in general this bijection is not a homomorphism for C^* -algebras of quasi-multipliers, so we can not exploit this bijection as in the previous remark.
- (4) Ideal hosts are of course more useful than hosts because then one has also unique extensions of representations on the same space. As we shall see however, their existence place strong structural restrictions on \mathfrak{S}_0 . Most of our analysis here will concern ideal hosts.

First we do a list of examples, and then some structural analysis.

- Examples** (1) For the pair $(\mathcal{F}, \mathfrak{S}_0) = (\mathcal{B}(\mathcal{H}), \mathfrak{S}_N)$ where \mathfrak{S}_N denotes the set of normal states, an ideal host algebra is $\mathcal{L} = \mathcal{K}(\mathcal{H})$ with the identity map $\xi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) = M(\mathcal{K}(\mathcal{H})) = \mathcal{E}$.
- (2) An important example for physics, is the following. Let the pair $(\mathcal{F}, \mathfrak{S}_0)$ consists of the CCR-algebra over a finite dimensional symplectic space, and its set of regular states. To be more concrete, consider the CCR-algebra on \mathbb{R}^2 , i.e. the unique simple C^* -algebra \mathcal{F} generated by unitaries $\{\delta_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{R}^2\}$ satisfying the Weyl relations:

$$\delta_{\mathbf{x}}\delta_{\mathbf{y}} = \rho(\mathbf{x}, \mathbf{y})\delta_{\mathbf{x}+\mathbf{y}}, \quad \text{where} \quad \rho(\mathbf{x}, \mathbf{y}) := \exp[i(x_1y_2 - x_2y_1)].$$

Define another C^* -algebra \mathcal{L} as the C^* -envelope of the twisted convolution algebra, where the latter consists of $L^1(\mathbb{R}^2)$ equipped with the multiplication and involution:

$$f * g(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}, \mathbf{x}) d\mathbf{y}, \quad f^*(\mathbf{x}) = \overline{f(-\mathbf{x})}.$$

This algebra \mathcal{L} is known to be isomorphic to $\mathcal{K}(L^2(\mathbb{R}))$ (cf. I.E. Segal [Se]). Then $\mathcal{F} \subset M(\mathcal{L})$ by the action $\delta_{\mathbf{x}} \cdot f(\mathbf{y}) = \rho(\mathbf{x}, \mathbf{y}) f(\mathbf{y} - \mathbf{x})$. The unique extensions of states on \mathcal{L} to \mathcal{F} produce precisely the set of regular states

$$\mathfrak{S}_0 := \{ \omega \in \mathfrak{S}(\mathcal{F}) \mid \mathbf{x} \rightarrow \omega(\delta_{\mathbf{x}}) \text{ is continuous} \}.$$

The extension map $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}_0$ is injective, since for each $\phi \in \mathfrak{S}_0$ we can reconstruct the $\omega \in \mathfrak{S}(\mathcal{L})$ such that $\theta(\omega) = \phi$ via the formula

$$\omega(f) := \int_{\mathbb{R}^2} f(\mathbf{x}) \varphi(\delta_{\mathbf{x}}) d\mathbf{x}$$

for all $f \in L^1(\mathbb{R}^2)$. (Put in other words, here \mathcal{F} is the twisted discrete group algebra for \mathbb{R}^2 , and \mathcal{L} is the usual twisted group algebra for \mathbb{R}^2 .) Since \mathcal{F} is simple, and $\mathfrak{S}_0 \neq \mathfrak{S}(\mathcal{F})$, we see that $\mathcal{F} \cap \mathcal{L} = \{0\}$. Of course \mathcal{L} is a far better behaved C^* -algebra than \mathcal{F} , it is even separable. See [Gr] for generalisations of this example.

- (3) Let G be a nondiscrete topological group, and denote G equipped with the discrete topology by G_d . Let the pair $(\mathcal{F}, \mathfrak{S}_0)$ be the discrete group algebra $\mathcal{F} = C^*(G_d)$ and its set of states continuous with respect to the topology of G (i.e. $g \rightarrow \omega(\delta_g)$ is continuous, where δ_g denotes the Dirac point measure at g). When G is locally compact, the usual group algebra $C^*(G)$ is an ideal host algebra if we use the imbedding $C^*(G_d) \subset M(C^*(G))$ obtained by convolution of measures.

Inspired by this example, we define

Def. Let G be a topological group, not necessarily locally compact. Then a *group algebra* for it, is any ideal host for the pair $(C^*(G_d), \mathfrak{S}_0)$ where \mathfrak{S}_0 denotes the states ω on $C^*(G_d)$ such that the map $g \rightarrow \omega(\delta_g)$ is continuous.

In [Gr] we took this definition for a group algebra in order to construct a group algebra for inductive limit groups. We can also adapt it for covariance algebras. In this paper, however, we will not construct any such group algebras for more general groups.

Next we analyze some structural consequences for the existence of a host for the pair $(\mathcal{F}, \mathfrak{S}_0)$. We will usually consider the homomorphism $\xi : \mathcal{F} \rightarrow \mathcal{E}$ as an embedding, to save on notation.

Theorem 1.1. *If a pair $(\mathcal{F}, \mathfrak{S}_0)$ has a host $\mathcal{L} \subset \mathcal{E}$, then*

- (i) \mathfrak{S}_0 is a norm-closed face in $\mathfrak{S}(\mathcal{F})$.
- (ii) If \mathcal{L} is an ideal host, then the norm-closed face \mathfrak{S}_0 is also invariant, i.e. if $\omega \in \mathfrak{S}_0$ then $\omega_B \in \mathfrak{S}_0$ for all $B \in \mathcal{F}$ with $\omega(B^*B) = 1$, and where $\omega_B(F) := \omega(B^*FB)$ for $F \in \mathcal{F}$.
- (iii) $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}_0$ is an isomorphism, i.e. it is affine and a homeomorphism w.r.t. the norm topology.
- (iv) In the case that \mathfrak{S}_0 is the face obtained by extending the states from a hereditary subalgebra $\mathcal{A} \subset \mathcal{F}$ to \mathcal{F} , then $\mathcal{L} \cap \mathcal{F} = \mathcal{A}$.

Proof: (i) Norm closure: We first show that $\theta : \mathcal{L}^* \rightarrow \mathcal{F}^*$ is norm continuous. Recall that since \mathcal{L} is hereditary in \mathcal{E} we have $\theta(\varphi)(F) := \varphi(PFP)$ with $P \in \mathcal{L}'' \subset \mathcal{E}'' \supset \mathcal{F}''$. Now

$$\begin{aligned} \|\theta(\varphi)\| &= \sup \{ |\theta(\varphi)(F)| \mid F \in \mathcal{F}, \|F\| \leq 1 \} \\ &= \sup \{ |\varphi(PFP)| \mid F \in \mathcal{F}, \|F\| \leq 1 \} \end{aligned}$$

Now we know from Kusuda [Ku] Theorem 2.2 that $P\mathcal{E}''P = \mathcal{L}''$ since \mathcal{L} is hereditary, and so

$$\{ |\varphi(PFP)| \mid F \in \mathcal{F}, \|F\| \leq 1 \} \subseteq \{ |\varphi(A)| \mid A \in \mathcal{L}'', \|A\| \leq 1 \}$$

and hence, since the supremum of the last set is just $\|\varphi\|$, we find that $\|\theta(\varphi)\| \leq \|\varphi\|$. Thus $\theta : \mathcal{L}^* \rightarrow \mathcal{F}^*$ is norm continuous. By assumption θ

is injective on $\mathfrak{S}(\mathcal{L})$; we prove that it is also injective on \mathcal{L}^* . If it were not, there would be $\varphi_i \in \mathcal{L}^*$ such that $\theta(\varphi_1 - \varphi_2) = 0$. Then for $\psi := \varphi_1 - \varphi_2$ do a Jordan decomposition, $\psi = \rho_+ - \rho_- + i(\mu_+ - \mu_-)$ and then $\theta(\psi) = 0$ implies $\theta(\rho_+ - \rho_-) = 0 = \theta(\mu_+ - \mu_-)$, i.e. $\theta(\rho_+) = \theta(\rho_-)$ and $\theta(\mu_+) = \theta(\mu_-)$. But θ is injective on states, so $\rho_+ = \rho_-$ and $\mu_+ = \mu_-$, i.e. $\psi = 0$. Thus we know that θ is both norm continuous and invertible on \mathcal{L}^* so by a corollary to the open mapping theorem, its inverse must also be norm continuous, and hence by Theorem 5.8, p216 of A. Taylor [Ta] we conclude that $\theta(\mathcal{L}^*)$ is norm closed. Since $\mathfrak{S}(\mathcal{F}) \subset \mathcal{F}^*$ is also norm closed (recall that \mathcal{F} is unital), so is $\theta(\mathcal{L}^*) \cap \mathfrak{S}(\mathcal{F})$ and as this is just the image under θ of $\mathfrak{S}(\mathcal{L})$, we conclude that \mathfrak{S}_0 is norm closed. Now (iii) also follows from the preceding.

That \mathfrak{S}_0 is a convex set, follows from the fact that θ is linear and $\mathfrak{S}(\mathcal{L})$ is convex, so we just need to prove the facial property. Let $\omega = \lambda\varphi + (1-\lambda)\psi \in \mathfrak{S}_0$ where $\lambda \in [0, 1]$. We need to show that $\varphi, \psi \in \mathfrak{S}_0$. Since $\omega \in \mathfrak{S}_0$ there is a unique $\omega' \in \mathfrak{S}(\mathcal{L})$ such that $\omega = \theta(\omega')$. By the hereditary property, ω' extends uniquely to a state $\widehat{\omega}'$ on \mathcal{E}'' (and this extension of course restricts to ω on \mathcal{F}). By definition of P we have that $\widehat{\omega}'(P) = 1$, and conversely, given any state γ on \mathcal{E}'' with $\gamma(P) = 1$ we have that $\gamma \upharpoonright \mathcal{L} \in \mathfrak{S}(\mathcal{L})$, hence $\theta(\gamma \upharpoonright \mathcal{L}) = \gamma \upharpoonright \mathcal{F} \in \mathfrak{S}_0$. Now $0 = \widehat{\omega}'(\mathbb{I} - P) = \lambda\widehat{\varphi}(\mathbb{I} - P) + (1-\lambda)\widehat{\psi}(\mathbb{I} - P)$ where $\widehat{\varphi}, \widehat{\psi}$ are extensions of φ, ψ to \mathcal{E}'' . Thus by positivity of all terms in the sum, we conclude that $\widehat{\varphi}, \widehat{\psi}$ vanish on $\mathbb{I} - P$ and hence $\varphi = \theta(\widehat{\varphi} \upharpoonright \mathcal{L})$, $\psi = \theta(\widehat{\psi} \upharpoonright \mathcal{L})$. Thus $\varphi, \psi \in \mathfrak{S}_0$, i.e. \mathfrak{S}_0 is a face.

For (ii), assume that \mathcal{L} is an ideal of \mathcal{E} , then we want to prove invariance of the face. Let $\omega = \theta(\omega') \in \mathfrak{S}_0$, then clearly $\omega'_B(\cdot) := \omega'(B^* \cdot B)$ defines a state on \mathcal{L} using the fact that \mathcal{L} is an ideal (here we took $B \in \mathcal{E}$ with $\omega'(B^*B) = 1$). Thus $\mathfrak{S}(\mathcal{L})$ is invariant, and by the definitions $\omega_B = \theta(\omega'_B) \in \mathfrak{S}_0$ because extension and conjugation of a state commutes when the projection P in Equation (1) commutes with \mathcal{F} , and this it does since \mathcal{L} is an ideal.

(iv) Since \mathcal{L} is hereditary in \mathcal{E} , we see that $\mathcal{L} \cap \mathcal{F}$ is hereditary in \mathcal{F} . Moreover, the face $\mathfrak{S}_0 \subset \mathfrak{S}(\mathcal{F})$ is precisely the states which extend from the states on \mathcal{L} , which are uniquely determined by their values on $\mathcal{L} \cap \mathcal{F}$ by the hereditary property. Since \mathfrak{S}_0 are also the states which extend from \mathcal{A} , by the bijection between such faces and hereditary subalgebras (cf. Pedersen 3.10.7 [Pe] and Murphy 3.2.1 [Mu]) we conclude that $\mathcal{L} \cap \mathcal{F} = \mathcal{A}$. ■

Remarks. (1) It is known that an invariant convex norm-closed set of states \mathfrak{S}_0 is also a face (cf. footnote in [HKK]), so for an ideal host it suffices to say that \mathfrak{S}_0 is an invariant convex norm-closed set. In other words, this says that the cone which \mathfrak{S}_0 generates, $\mathbb{R}_+\mathfrak{S}_0$, is a folium. We will extend the term “folium” to also mean invariant norm-closed convex sets of state spaces.

(2) Since $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}_0$ is an affine bijection, it restricts to a bijection between the pure states on \mathcal{L} and the extreme points of \mathfrak{S}_0 . This immediately limits the class of faces and folia for which hosts exist, because there are many folia without extreme points, e.g. the folium of normal states of $L^\infty(\mathbb{R})$ (measures absolutely continuous w.r.t. the Lebesgue measure). We will sharpen this observation below.

Note also that as θ involves both an extension and a restriction, it need not a priori take pure states to pure states. Since $\mathfrak{S}(\mathcal{L})$ is generated as the w^* -closed convex hull of its pure states, we can use θ to transfer this weak*-topology to \mathfrak{S}_0 (but this is different from the weak*-topology of \mathcal{F}), to conclude that w.r.t. this topology \mathfrak{S}_0 is a compact convex set, hence via Choquet theory, there are integral decompositions of states w.r.t. measures on \mathfrak{S}_0 . In the next section we will exploit these decompositions. It would be nice to have some intrinsic definition of the w^* -topology induced by θ on \mathfrak{S}_0 but we do not have this yet.

Theorem 1.2. *Let $\mathfrak{f} \subset (\mathcal{F}^*)_+$ be a folium. Then it is the set of normal positive forms of the von Neumann algebra $\pi(\mathcal{F})''$ where $\pi = \bigoplus_{\varphi \in \mathfrak{f}} \pi_\varphi$. Conversely, the set of positive normal forms of any von Neumann algebra is a folium.*

Proof: See Haag, Kadison, Kastler in [HKK]. ■

This is quite useful, in that any folium can now be analyzed as the normal state space of some concrete C^* -algebra.

Remark 1.3. For later use, we need to know about projections associated with faces and folia. Start with a pair $(\mathcal{F}, \mathfrak{S}_0)$, where by Theorem 1.1 we now assume that \mathfrak{S}_0 is a norm-closed face. Corresponding to this, we know from Pedersen 3.6.11 [Pe] that there is a projection $P \in \mathcal{F}''$ which we now show how to construct. (To use Pedersen 3.6.11, we need to know that a norm closed face \mathfrak{S}_0 generates a cone $\mathbb{R}_+\mathfrak{S}_0$ which is norm-closed and hereditary and in $(\mathcal{F}'')_*$, but it is quite straightforward to verify this). First define

$$\Gamma := \{ \varphi \in (\mathcal{F}'')_* \mid |\varphi| \in \mathbb{R}_+\mathfrak{S}_0 \},$$

and this is in fact a left invariant vector space, by the proof in Pedersen 3.6.11. Then its annihilator $\Gamma^\perp \subset \mathcal{F}''$ is a σ -weakly closed left ideal, hence $\Gamma^\perp \cap (\Gamma^\perp)^\perp$ is a weak-operator closed hereditary subalgebra of \mathcal{F}'' . If we denote its unit (which is a projection in \mathcal{F}'') by Q , then the desired projection we want is $P = \mathbb{I} - Q$. To recover $\mathbb{R}_+\mathfrak{S}_0$ from P , we just take the set of $\varphi \in (\mathcal{F}''_*)_+ = (\mathcal{F}^*)_+$ such that $\varphi(P) = 1$.

In the case that \mathfrak{S}_0 is a folium (i.e. also invariant), we find that Γ is a two-sided invariant space, hence Γ^\perp is a two-sided σ -weakly closed ideal. It then follows from Pedersen 2.5.4 [Pe] that its unit $Q \in \mathcal{F}' \cap \mathcal{F}''$, and hence $P \in \mathcal{F}' \cap \mathcal{F}''$.

An obvious method by which one may think one can construct a host algebra, is to take the algebra $\tilde{\mathcal{L}} := C^*(P\mathcal{F}P) \subset \mathcal{F}''$. Whilst this is certainly hereditary in \mathcal{F}'' , and the states which uniquely extend from $\tilde{\mathcal{L}}$ to $\mathcal{E} := C^*(\tilde{\mathcal{L}} \cup \mathcal{F}) \subset \mathcal{F}''$ will satisfy $\omega(P) = 1$, this is not enough to guarantee that their restrictions to \mathcal{F} will be in \mathfrak{S}_0 . This is because given a $\varphi \in \mathfrak{S}(\mathcal{F})$, one can only conclude that $\varphi \in \mathfrak{S}_0$ if its *normal* extension to \mathcal{F}'' satisfies $\tilde{\varphi}(P) = 1$, and whilst for an $\omega \in \mathfrak{S}(\mathcal{E})$ which extended from one on $\tilde{\mathcal{L}}$ we have $\omega(P) = 1$, we do not know that ω is the normal extension of its restriction $\omega \upharpoonright \mathcal{F}$. Thus θ may not map onto \mathfrak{S}_0 for this choice $\tilde{\mathcal{L}}$. By the previous remark we know there are folia without hosts, so that we know the above procedure must sometimes fail.

From Takesaki Prop. 2.17 (p129) [Tak] we know that if \mathcal{I} is a closed two-sided ideal of \mathcal{A} , then $\pi(\mathcal{I})'' = \pi(\mathcal{A})''$ for any representation π which is nondegenerate on \mathcal{I} .

This implies that if we have an ideal host $\mathcal{L} \subset \mathcal{E}$ for $(\mathcal{F}, \mathfrak{S}_0)$, then $\pi(\mathcal{L})'' = \pi(\mathcal{E})''$ for the representation $\pi = \bigoplus_{\omega \in \mathfrak{S}(\mathcal{L})} \pi_\omega$. This fact leads us to suspect that $\pi(\mathcal{L})'' = \pi(\mathcal{F})''$

and this is what we now want to prove, but we need a lemma first. We use Pedersen's notation $[\cdot]$ for “closed linear span.”

Lemma 1.4. *Let $(\mathcal{F}, \mathfrak{S}_0)$ have an ideal host $\mathcal{L} \subset \mathcal{E}$, and let $\pi : \mathcal{E} \rightarrow \mathcal{B}(\mathcal{H})$ be a representation. Let \mathcal{H}_e be the essential subspace of $\pi(\mathcal{L})$. Then for any vector $\Omega \in \mathcal{H}_e$ we have*

$$[\pi(\mathcal{L})\Omega] = [\pi(\mathcal{F})\Omega].$$

Proof: Denote $\mathcal{H}_\Omega := [\pi(\mathcal{L})\Omega]$ and $\pi_\Omega := \pi \upharpoonright \mathcal{H}_\Omega$, then due to the fact that \mathcal{L} is an ideal host, π_Ω extends uniquely on the same space \mathcal{H}_Ω to \mathcal{F} . Hence $[\pi(\mathcal{F})\Omega] = [\pi_\Omega(\mathcal{F})\Omega] \subset \mathcal{H}_\Omega = [\pi(\mathcal{L})\Omega]$. We prove the reverse inclusion by contradiction. Assume it is not true, then there exists some nonzero $\psi \in [\pi(\mathcal{L})\Omega] \subset \mathcal{H}_e$ such that $\psi \perp [\pi(\mathcal{F})\Omega]$. We have

$$\pi(\mathcal{F})\psi \perp \pi(\mathcal{F})\Omega \quad (5)$$

because $(\pi(A)\psi, \pi(B)\Omega) = (\psi, \pi(A^*B)\Omega) = 0$ for all $A, B \in \mathcal{F}$. Normalise: $\|\psi\| = 1 = \|\Omega\|$ and choose $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha \neq 0 \neq \beta$ and define:

$$\begin{aligned} \varphi &:= \alpha\Omega + \beta\psi, & \omega_\varphi(A) &:= (\varphi, \pi(A)\varphi) \\ \omega_\psi(A) &:= (\psi, \pi(A)\psi), & \omega(A) &:= (\Omega, \pi(A)\Omega). \end{aligned}$$

Now since $\varphi \in \mathcal{H}_e$, ω_φ is nondegenerate on \mathcal{L} , hence $\theta(\omega_\varphi \upharpoonright \mathcal{L}) = \omega_\varphi \upharpoonright \mathcal{F}$ and so for all $A \in \mathcal{F}$ we have:

$$\begin{aligned} \theta(\omega_\varphi)(A) &= \omega_\varphi(A) = (\alpha\Omega + \beta\psi, \pi(A)(\alpha\Omega + \beta\psi)) \\ &= |\alpha|^2\omega(A) + |\beta|^2\omega_\psi(A) = \lambda\omega(A) + (1 - \lambda)\omega_\psi(A) \end{aligned}$$

where we made use of the orthogonality (5) and we have set $\lambda := |\alpha|^2 \in (0, 1)$. Thus

$$\theta(\omega_\varphi) = \theta(\lambda\omega + (1 - \lambda)\omega_\psi). \quad (6)$$

However, for an element $L \in \mathcal{L}$ we cannot use the orthogonality (5), and so we get

$$\omega_\varphi(L) = \lambda\omega(L) + (1 - \lambda)\omega_\psi(L) + \bar{\alpha}\beta(\Omega, \pi(L)\psi) + \alpha\bar{\beta}(\psi, \pi(L)\Omega) \quad (7)$$

We show that the last two terms can always be made nonzero by some choice of L . If this were not the case, they must be zero for all $L = \gamma R$ where $\gamma \in \mathbb{C}$, $R = R^* \in \mathcal{L}$. Then we have for the last two terms of Equation (7) that

$$2\operatorname{Re}[\alpha\bar{\beta}\gamma(\psi, \pi(R)\Omega)] = 0$$

for all γ and R , i.e. $(\psi, \pi(R)\Omega) = 0$ for all $R = R^* \in \mathcal{L}$. However, \mathcal{L} is spanned by its selfadjoint elements, thus $\psi \perp \pi(\mathcal{L})\Omega$, and so

$$\psi \perp [\pi(\mathcal{L})\Omega] \ni \psi$$

and thus $\psi = 0$ which is a contradiction with our initial assumption. Thus the last two terms of Equation (7) are nonzero for some L , i.e. on \mathcal{L} we have

$$\omega_\varphi \neq \lambda\omega + (1 - \lambda)\omega_\psi.$$

This, together with Equation (6) contradicts the assumption that θ is injective on $\mathfrak{S}(\mathcal{L})$. Thus our initial assumption is wrong, so

$$[\pi(\mathcal{L})\Omega] \subseteq [\pi(\mathcal{F})\Omega], \text{ and in fact we have equality } [\pi(\mathcal{L})\Omega] = [\pi(\mathcal{F})\Omega]. \quad \blacksquare$$

Remarks. (1) Until now, we have used the standard notation \mathcal{F}'' for the universal von Neumann algebra of \mathcal{F} (not unique for concrete C^* -algebras). To avoid confusion in subsequent arguments, we will sometimes explicitly indicate the universal representations, and our notation is that $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{B}(\mathcal{H}_{\mathcal{F}})$ is the universal representation of \mathcal{F} , i.e. $\pi_{\mathcal{F}} = \bigoplus_{\omega \in \mathfrak{S}(\mathcal{F})} \pi_\omega$ and $\mathcal{F}'' \equiv \pi_{\mathcal{F}}(\mathcal{F})''$. Note that when \mathcal{L} is an ideal host

for $(\mathcal{F}, \mathfrak{S}_0)$, then $\pi_{\mathcal{L}}$ extends uniquely on the same space to a representation of \mathcal{F} , and this implies that $\pi_{\mathcal{L}}$ of \mathcal{F} is a subrepresentation of $\pi_{\mathcal{F}}$.

(2) One may try to generalise this lemma away from ideal hosts to hosts, in which case we suspect that for any vector $\Omega \in \mathcal{H}_e$ we have

$$[\pi(\mathcal{L})\Omega] \subseteq [\pi(\mathcal{F})\Omega].$$

However, the proof so far eludes us. If one starts as in the proof by assuming some nonzero $\psi \in [\pi(\mathcal{L})\Omega] \setminus [\pi(\mathcal{F})\Omega]$, then $\psi = \psi_0 + \psi_1$ where $\psi_0 \in [\pi(\mathcal{F})\Omega] \perp \psi_1$ but we may have that $\psi_i \notin \mathcal{H}_e$, even though $\psi \in \mathcal{H}_e$, and this causes problems. Now we are ready to prove:

Theorem 1.5. *Let $(\mathcal{F}, \mathfrak{S}_0)$ have an ideal host \mathcal{L} , then $\pi_{\mathcal{L}}(\mathcal{F})'' = \pi_{\mathcal{L}}(\mathcal{L})'' (\equiv \mathcal{L}'')$ where we use the same symbol $\pi_{\mathcal{L}}$ for the unique extension of it from \mathcal{L} to $\mathcal{L}'' \supset M(\mathcal{L}) \supset \mathcal{F}$.*

Proof: In $\mathcal{B}(\mathcal{H}_{\mathcal{L}})$ let $A \in \pi_{\mathcal{L}}(\mathcal{L})'$ and let $B \in \mathcal{F}$, and recall that $\pi_{\mathcal{L}}(B)\psi = \lim_{\alpha} \pi_{\mathcal{L}}(BE_{\alpha})\psi$ for all $\psi \in \mathcal{H}_{\mathcal{L}}$ and any approximate identity $\{E_{\alpha}\}$ of \mathcal{L} . Thus for all $\psi \in \mathcal{H}_{\mathcal{L}}$ we have

$$[A, \pi_{\mathcal{L}}(B)]\psi = \lim_{\alpha} [A, \pi_{\mathcal{L}}(BE_{\alpha})]\psi = 0$$

because $A \in \pi_{\mathcal{L}}(\mathcal{L})'$ and \mathcal{L} is an ideal. This is true for all $B \in \mathcal{F}$, and so $\pi_{\mathcal{L}}(\mathcal{L})' \subseteq \pi_{\mathcal{L}}(\mathcal{F})'$.

We now prove the reverse inclusion, and we do it by contradiction. Assume that $\pi_{\mathcal{L}}(\mathcal{F})' \neq \pi_{\mathcal{L}}(\mathcal{L})'$. Then since von Neumann algebras are spanned by their projections, we can find a nontrivial projection $P \in \pi_{\mathcal{L}}(\mathcal{F})' \setminus \pi_{\mathcal{L}}(\mathcal{L})'$ (otherwise,

if all the projections of $\pi_{\mathcal{L}}(\mathcal{F})'$ were in $\pi_{\mathcal{L}}(\mathcal{L})'$ the algebras would be equal). Recall that $\pi_{\mathcal{L}} = \bigoplus_{\omega \in \mathfrak{S}(\mathcal{L})} \pi_{\omega}$, so there must be some state in $\mathfrak{S}(\mathcal{L})$, say ω_0 , such that

$$[\pi_{\mathcal{L}}(\mathcal{L}), P] P_{\omega_0} \neq 0 = [\pi_{\mathcal{L}}(\mathcal{F}), P] \quad (8)$$

where P_{ω_0} denotes the projection onto the subspace $\mathcal{H}_{\omega_0} \subset \mathcal{H}_{\mathcal{L}}$ of the subrepresentation $\pi_{\omega_0} : \mathcal{L} \rightarrow \mathcal{B}(\mathcal{H}_{\omega_0})$. Let Ω_{ω_0} be the normalised cyclic vector for this representation. We claim that

$$\|P\Omega_{\omega_0}\| \neq 0 \neq \|(\mathbb{I} - P)\Omega_{\omega_0}\|.$$

If $P\Omega_{\omega_0} = 0$, then

$$0 = [\pi_{\mathcal{L}}(\mathcal{F})P\Omega_{\omega_0}] = [P\pi_{\mathcal{L}}(\mathcal{F})\Omega_{\omega_0}] = [P\pi_{\omega_0}(\mathcal{F})\Omega_{\omega_0}] = P\mathcal{H}_{\omega_0}$$

where we made use of Lemma 1.4, that Ω_{ω_0} is cyclic for $\pi_{\omega_0}(\mathcal{F})$. But if P annihilates \mathcal{H}_{ω_0} , it must commute with $\pi_{\mathcal{L}}(\mathcal{L})$ on \mathcal{H}_{ω_0} , and this contradicts Equation (8), hence $P\Omega_{\omega_0} \neq 0$.

Similarly, if $(\mathbb{I} - P)\Omega_{\omega_0} = 0$, then by the same argument $(\mathbb{I} - P)$ commutes with $\pi_{\mathcal{L}}(\mathcal{L})$ on \mathcal{H}_{ω_0} , hence so does P . So, also $(\mathbb{I} - P)\Omega_{\omega_0} \neq 0$. Thus we can write:

$$\begin{aligned} \Omega_{\omega_0} &= P\Omega_{\omega_0} + (\mathbb{I} - P)\Omega_{\omega_0} \\ &= \alpha \cdot \Omega_P + \beta \cdot \Omega_{P^\perp} \quad \text{where:} \\ \Omega_P &:= \frac{P\Omega_{\omega_0}}{\|P\Omega_{\omega_0}\|}, \quad \Omega_{P^\perp} := \frac{(\mathbb{I} - P)\Omega_{\omega_0}}{\|(\mathbb{I} - P)\Omega_{\omega_0}\|} \end{aligned}$$

and where $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha \neq 0 \neq \beta$. Adapting now the proof in Lemma 1.4, since we have that $[\pi_{\mathcal{L}}(\mathcal{F}), P] = 0$ we find for all $A \in \mathcal{F}$:

$$\begin{aligned} \omega_0(A) &= \lambda \omega_P(A) + (1 - \lambda) \omega_{P^\perp}(A) \quad \text{where:} \\ \omega_P(N) &= (\Omega_P, \pi_{\mathcal{L}}(N)\Omega_P), \quad \text{and} \\ \omega_{P^\perp}(N) &= (\Omega_{P^\perp}, \pi_{\mathcal{L}}(N)\Omega_{P^\perp}), \end{aligned}$$

and we set $\lambda := |\alpha|^2$. That is, we have

$$\theta(\omega_0) = \theta(\lambda \omega_P + (1 - \lambda) \omega_{P^\perp}). \quad (9)$$

Now for an $L \in \mathcal{L}$ we have similar to before:

$$\begin{aligned} \omega_0(L) &= (\alpha \Omega_P + \beta \Omega_{P^\perp}, \pi_{\mathcal{L}}(L)(\alpha \Omega_P + \beta \Omega_{P^\perp})) \\ &= \lambda \omega_P(L) + (1 - \lambda) \omega_{P^\perp}(L) \\ &\quad + \bar{\alpha} \beta (\Omega_P, \pi_{\mathcal{L}}(L)\Omega_{P^\perp}) + \alpha \bar{\beta} (\Omega_{P^\perp}, \pi_{\mathcal{L}}(L)\Omega_P). \end{aligned} \quad (10)$$

We prove that we can always find an $L \in \mathcal{L}$ to make the last line nonzero. If it is always zero, it is zero for all $L = \gamma R$ where $\gamma \in \mathbb{C}$ and $R = R^* \in \mathcal{L}$. Thus

$$2\operatorname{Re}[\alpha\bar{\beta}\gamma(\Omega_{P^\perp}, \pi_{\mathcal{L}}(R)\Omega_P)] = 0 \quad \forall \gamma \in \mathbb{C}, R = R^* \in \mathcal{L}$$

and thus $(\Omega_{P^\perp}, \pi_{\mathcal{L}}(R)\Omega_P) = 0$ for all $R = R^* \in \mathcal{L}$. But \mathcal{L} is spanned by its selfadjoint elements, and so

$$(\Omega_{P^\perp}, \pi_{\mathcal{L}}(\mathcal{L})\Omega_P) = 0 \quad \text{i.e.} \quad \pi_{\mathcal{L}}(\mathcal{L})\Omega_P \perp \pi_{\mathcal{L}}(\mathcal{L})\Omega_{P^\perp}.$$

Thus $\pi_{\mathcal{L}}(\mathcal{L})$ restricted to $[\pi_{\mathcal{L}}(\mathcal{L})\Omega_P] \oplus [\pi_{\mathcal{L}}(\mathcal{L})\Omega_{P^\perp}]$ decomposes into two cyclic representations $\pi_P \oplus \pi_{P^\perp}$. Since Ω_P is cyclic for $\pi_P(\mathcal{L})$, by Lemma 1.4 it is also cyclic for $\pi_P(\mathcal{F})$. Thus,

$$[\pi_{\mathcal{L}}(\mathcal{L})\Omega_P] = [\pi_{\mathcal{L}}(\mathcal{F})\Omega_P] = [\pi_{\mathcal{L}}(\mathcal{F})P\Omega] = [P\pi_{\mathcal{L}}(\mathcal{F})\Omega] = P\mathcal{H}_{\omega_0}$$

and likewise we get $[\pi_{\mathcal{L}}(\mathcal{L})\Omega_{P^\perp}] = (\mathbb{I} - P)\mathcal{H}_{\omega_0}$. So for all $L \in \mathcal{L}$ and $A \in \mathcal{F}$ we have

$$\begin{aligned} P\pi_{\mathcal{L}}(L)\pi_{\mathcal{L}}(A)\Omega &= P\pi_{\mathcal{L}}(LA)(\alpha\Omega_P + \beta\Omega_{P^\perp}) = \alpha\pi_{\mathcal{L}}(LA)\Omega_P \\ &= \pi_{\mathcal{L}}(LA)P\Omega = \pi_{\mathcal{L}}(L)P\pi_{\mathcal{L}}(A)\Omega. \end{aligned}$$

Thus $[P, \pi_{\mathcal{L}}(L)]P\omega_0 = 0$ for all $L \in \mathcal{L}$. But this contradicts Equation (8), hence the last line in Equation (10) is nonzero for some $L \in \mathcal{L}$. Thus

$$\omega_0 \neq \lambda\omega_P + (1 - \lambda)\omega_{P^\perp}$$

and this, together with Equation (9) now contradicts the assumption that θ is injective on $\mathfrak{S}(\mathcal{L})$. Thus, the initial assumption was wrong, and we conclude $\pi_{\mathcal{L}}(\mathcal{L})' = \pi_{\mathcal{L}}(\mathcal{F})'$, and hence $\pi_{\mathcal{L}}(\mathcal{L})'' = \pi_{\mathcal{L}}(\mathcal{F})''$. ■

The unique extension of $\pi_{\mathcal{L}}$ from \mathcal{L} to \mathcal{F} is of course just the representation $\pi_{\mathfrak{S}_0} := \bigoplus_{\omega \in \mathfrak{S}_0} \pi_\omega \in \operatorname{Rep}(\mathcal{F})$, using the definition of an ideal host.

Corollary 1.6. *Let \mathcal{N} be a von Neumann algebra, and let \mathfrak{S}_0 be its set of normal states. If \mathcal{L} is an ideal host for the pair $(\mathcal{N}, \mathfrak{S}_0)$, then $\pi_{\mathfrak{S}_0}(\mathcal{N}) = \mathcal{L}''$ and hence $\pi_{\mathfrak{S}_0}(\mathcal{N})$ contains \mathcal{L} as an ideal.*

Proof: Recall the embedding $\mathcal{N} \subset M(\mathcal{L}) \subset \mathcal{L}''$. Note that if $\omega \in \mathfrak{S}(\mathcal{L})$, then the unique extension of π_ω to \mathcal{N} is (unitarily equivalent to) $\pi_{\theta(\omega)}$, and this we see from

$$\theta(\omega)(N) = \lim_{\alpha} \omega(NE_{\alpha}) = \lim_{\alpha} (\Omega_{\omega}, \pi_{\omega}(NE_{\alpha})\Omega_{\omega}) = (\Omega_{\omega}, \tilde{\pi}_{\omega}(N)\Omega_{\omega}).$$

Thus, recalling that the universal representation of \mathcal{L} is $\pi_{\mathcal{L}} = \bigoplus_{\omega \in \mathfrak{S}(\mathcal{L})} \pi_{\omega}$ and that $\theta(\mathfrak{S}(\mathcal{L})) = \mathfrak{S}_0$, we conclude that the unique extension of $\pi_{\mathcal{L}}$ to \mathcal{N} is

$\pi_{\mathcal{L}} \upharpoonright \mathcal{N} = \bigoplus_{\omega \in \mathfrak{S}_0} \pi_{\omega} = \pi_{\mathfrak{S}_0}$. Thus $\pi_{\mathfrak{S}_0}$ on \mathcal{N} is a normal representation, and so

$$\pi_{\mathfrak{S}_0}(\mathcal{N}) = \pi_{\mathfrak{S}_0}(\mathcal{N}'') = \pi_{\mathfrak{S}_0}(\mathcal{N})'' = \mathcal{L}''$$

where we used Theorem 1.5 for the last equality. ■

Corollary 1.7. *If \mathcal{L} is an ideal host for a pair $(\mathcal{F}, \mathfrak{S}_0)$, then $\theta : \mathfrak{S}(\mathcal{L}) \rightarrow \mathfrak{S}_0$ maps the pure states of \mathcal{L} to pure states on \mathcal{F} . Hence all extreme points of \mathfrak{S}_0 are pure.*

Proof: By Theorem 1.5 we have $\pi_{\mathfrak{S}_0}(\mathcal{F})'' = \pi_{\mathfrak{S}_0}(\mathcal{L})''$. Let $\omega_0 \in \mathfrak{S}_0$, then since $\pi_{\mathfrak{S}_0} = \bigoplus_{\omega \in \mathfrak{S}_0} \pi_{\omega}$, we have that the restriction map $R : \pi_{\mathfrak{S}_0}(\mathcal{F}) \rightarrow \pi_{\omega_0}(\mathcal{F})$ by $R(\pi_{\mathfrak{S}_0}(F)) := \pi_{\omega_0}(F)$ for all $F \in \mathcal{F}$ is a normal $*$ -homomorphism, hence it extends to $\pi_{\mathfrak{S}_0}(\mathcal{F})'' = \pi_{\mathcal{L}}(\mathcal{L})''$ and so

$$\begin{aligned} R(\pi_{\mathfrak{S}_0}(\mathcal{F})'') &= R(\pi_{\mathfrak{S}_0}(\mathcal{F}))'' = \pi_{\omega_0}(\mathcal{F})'' \\ &= R(\pi_{\mathcal{L}}(\mathcal{L}))'' = \pi_{\omega_0}(\mathcal{L})''. \end{aligned}$$

Or to be more notationally precise, $\pi_{\omega_0}(\mathcal{F})'' = \pi_{\theta^{-1}(\omega_0)}(\mathcal{L})''$. Now θ is an affine bijection, so it restricts to a bijection between the pure states of $\mathfrak{S}(\mathcal{L})$ and the extreme points of \mathfrak{S}_0 . If φ is a pure state on \mathcal{L} , then $\mathcal{B}(\mathcal{H}_{\varphi}) = \pi_{\varphi}(\mathcal{L})'' = \pi_{\theta(\varphi)}(\mathcal{F})''$, i.e. $\theta(\varphi)$ is also pure on \mathcal{F} . ■

This last corollary now severely limits the class of folia for which ideal hosts exist, indeed, we can quickly prove many obstruction theorems, e.g. the next one.

Corollary 1.8. *If \mathcal{N} is a simple factor and \mathfrak{S}_0 is its folium of normal states, then there is no ideal host for the pair $(\mathcal{N}, \mathfrak{S}_0)$.*

Proof: it suffices by Corollary 1.7 to observe that \mathfrak{S}_0 contains no pure states. For if an $\omega \in \mathfrak{S}_0$ were pure, then using the fact that π_{ω} is normal, we see that $\pi_{\omega}(\mathcal{N}) = \pi_{\omega}(\mathcal{N})'' = \mathcal{B}(\mathcal{H}_{\omega})$. Since \mathcal{N} is simple, π_{ω} is an isomorphism hence $\mathcal{N} \cong \mathcal{B}(\mathcal{H}_{\omega})$ and the latter is not simple. This is a contradiction, so \mathfrak{S}_0 has no pure states. ■

Remarks. (1) Thus, since we know from Kadison and Ringrose 6.8.4 and 6.6.5 [KR], that every finite factor and each countably decomposable type III factor is simple, Corollary 1.8 shows that there are many von Neumann algebras for which there is no ideal host for its normal states. From Corollary 1.6 we see that we can only expect ideal hosts for a von Neumann algebra which has a norm closed proper ideal which is weak operator dense. If \mathcal{F} is a C^* -algebra, but not a von Neumann algebra, Theorem 1.5 just tells us that we should be looking for an ideal host \mathcal{L} in the von Neumann algebra $\pi_{\mathfrak{S}_0}(\mathcal{F})''$, but by Corollary 1.7 we also know that this may fail, unless all the extreme points of \mathfrak{S}_0 are pure.

(2) Theorem 1.5 states a “weak” uniqueness, in that it claims that all ideal hosts for the same pair must have the same universal algebra

$$\mathcal{L}'' = \pi_{\mathfrak{S}_0}(\mathcal{F})''.$$

Theorem 1.5 also tells us where all ideal hosts reside, viz $\mathcal{L} \subset \pi_{\mathfrak{S}_0}(\mathcal{F})'' \subset \mathcal{B}(\mathcal{H}_{\mathfrak{S}_0})$, and we can make this even more precise: Since \mathcal{L} is an ideal host, \mathfrak{S}_0 is a folium, and so for the projection associated with \mathfrak{S}_0 we have $P \in \mathcal{F}' \cap \mathcal{F}''$ (cf. Remark 1.3) and $\mathfrak{S}_0 = \{\omega \in \mathfrak{S}(\mathcal{F}) \mid \tilde{\omega}(P) = 1\}$ where $\tilde{\omega}$ denotes the normal extension of ω from \mathcal{F} to \mathcal{F}'' . Thus in $\pi_{\mathcal{F}}$ we have $P\Omega_\omega = \Omega_\omega$ for all $\omega \in \mathfrak{S}_0$, hence by $P \in \mathcal{F}' \cap \mathcal{F}''$ we see $PF\Omega_\omega = F\Omega_\omega$ for all $F \in \mathcal{F}''$. Thus P is the projector onto $\bigoplus_{\omega \in \mathfrak{S}_0} \mathcal{H}_\omega = \mathcal{H}_{\mathfrak{S}_0} \subset \mathcal{H}_{\mathcal{F}}$, hence $P\pi_{\mathcal{F}}(\mathcal{F})'' = \pi_{\mathfrak{S}_0}(\mathcal{F})''$ and so we conclude from $P \in \mathcal{F}''$ that

$$\mathcal{L} \subset \pi_{\mathfrak{S}_0}(\mathcal{F})'' \subset \pi_{\mathcal{F}}(\mathcal{F})'' = \mathcal{F}''. \quad (11)$$

- (3) Since for an ideal host \mathcal{L} for a pair $(\mathcal{F}, \mathfrak{S}_0)$ we know by Theorem 1.5 that $\xi(\mathcal{F}) \subseteq M(\mathcal{L}) \subset \mathcal{L}'' = \pi_{\mathcal{L}}(\mathcal{F})''$, we conclude that the embedding $\xi : \mathcal{F} \rightarrow M(\mathcal{L})$ is precisely $\pi_{\mathfrak{S}_0} \upharpoonright \mathcal{F}$. So ξ is faithful iff $\pi_{\mathfrak{S}_0} \upharpoonright \mathcal{F}$ is faithful iff for each $F \in \mathcal{F}$ we have $\omega(F) \neq 0$ for some $\omega \in \mathfrak{S}_0$.
- (4) We can easily adapt the proofs of 1.4 and 1.5 to prove a Stone–Weierstrass theorem for von Neumann algebras, i.e. if a von Neumann algebra contains a sub-von Neumann algebra which separates its normal states, then they are equal. However, this fact has also a very short proof via the bipolar theorem (Private communication with R.Longo).
- (5) By Corollary 1.7 we can also find topological groups which have no group algebras (i.e. ideal hosts for the pair $(C^*(G_d), \mathfrak{S}_0)$ where \mathfrak{S}_0 are states ω for which $g \rightarrow \omega(\delta_g)$ is continuous). For example, let $G = L^\infty(\mathbb{R})$ with the group operation being addition, and with the strong operator topology w.r.t. its representation as multiplication operators on $L^2(\mathbb{R})$. Then there is a topological isomorphism $L^\infty(\mathbb{R}) \cong C(\mathbb{R}_s)$ where \mathbb{R}_s is \mathbb{R} compactified and endowed with a suitable hyperstonean topology (cf. proof of Theorem III.1.18 [Tak], or Theorem 2.1 below). Any irreducible representation of G must be a character, hence point evaluation on $C(\mathbb{R}_s)$, and this cannot be continuous w.r.t. the strong operator topology, because points are still of measure zero w.r.t. the extension of the Lebesgue measure to \mathbb{R}_s . Thus the folium \mathfrak{S}_0 of states of $C^*(G_d)$ which are continuous w.r.t. the topology of G contains no pure states, hence by Corr. 1.7 we conclude that G has no group algebra.

To conclude this section, we would like to make precise the relation between the representations of a host \mathcal{L} for a pair $(\mathcal{F}, \mathfrak{S}_0)$ and the representations of \mathcal{F} . Denote the normal representations of $\pi_{\mathfrak{S}_0}(\mathcal{F})$ by $\text{Rep}_{\mathfrak{S}_0} \mathcal{F}$. By Theorem 1.5 we know that $\mathcal{L}'' \subseteq \pi_{\mathfrak{S}_0}(\mathcal{F})''$, hence for each $\pi \in \text{Rep}_{\mathfrak{S}_0} \mathcal{F}$ we can construct a representation $\Lambda(\pi) \in \text{Rep} \mathcal{L}$ by first extending π via strong operator continuity to a representation $\tilde{\pi} \in \text{Rep}(\pi_{\mathfrak{S}_0}(\mathcal{F})'')$, and then defining $\Lambda(\pi)$ as $\tilde{\pi} \upharpoonright \mathcal{L}$ restricted to its essential subspace. This produces a map $\Lambda : \text{Rep}_{\mathfrak{S}_0} \mathcal{F} \rightarrow \text{Rep} \mathcal{L}$.

Theorem 1.9. *If \mathcal{L} is an ideal host for the pair $(\mathcal{F}, \mathfrak{S}_0)$, then the map $\Lambda : \text{Rep}_{\mathfrak{S}_0} \mathcal{F} \rightarrow \text{Rep} \mathcal{L}$ is a bijection which takes irreducible representations to irreducible representations. In the case that \mathcal{L} is merely a host, Λ is a bijection modulo unitary equivalence. Its inverse is via inducing of representations; $\Lambda^{-1}(\{\pi\}) = \{\text{Ind}_{\mathcal{L}}^{\mathcal{F}}(\pi)\}$ where $\{\cdot\}$ denotes unitary*

equivalence classes, and the induction is done via the right \mathcal{L} -rigged left \mathcal{F} -module $\mathcal{M} := [\mathcal{FL}] \subset \mathcal{E}$ with rigging map $\langle u, v \rangle := u^*v \in \mathcal{L}$ for all $u, v \in \mathcal{M}$.

Proof: If \mathcal{L} is an ideal host, $\mathcal{L}'' = \pi_{\mathfrak{S}_0}(\mathcal{F})''$, and now as both \mathcal{L} and $\pi_{\mathfrak{S}_0}(\mathcal{F})$ are strong operator dense in \mathcal{L}'' , it is obvious that each uniquely determines a normal representation, and so the proof for this case ends here.

For the case of \mathcal{L} just a host, let $(\pi, \mathcal{H}) \in \text{Rep}_{\mathfrak{S}_0} \mathcal{F}$, so $\Lambda(\pi)$ is $\tilde{\pi} \upharpoonright \mathcal{L}$ restricted to its essential subspace $[\tilde{\pi}(\mathcal{L})\mathcal{H}]$. Now for our proof, we will first construct $\text{Ind}_{\mathcal{L}}^{\mathcal{F}}(\Lambda(\pi))$, show it is unitary equivalent to π , and then show that every $\pi \in \text{Rep}_{\mathfrak{S}_0} \mathcal{F}$ is unitarily equivalent to some $\text{Ind}_{\mathcal{L}}^{\mathcal{F}}(\gamma)$, $\gamma \in \text{Rep} \mathcal{L}$. Following Fell and Doran XI.4.12 [FD] or Rieffel [Ri], consider the right \mathcal{L} -rigged left \mathcal{F} -module

$$\mathcal{M} := [\mathcal{FL}] \subset \mathcal{E}, \quad \langle u, v \rangle := u^*v \in \mathcal{L} \quad \forall u, v \in \mathcal{M}$$

where we used the fact that \mathcal{L} is hereditary in \mathcal{E} to conclude $u^*v \in \mathcal{L}$. Since \mathcal{L} is a C^* -algebra, every representation of it is inducible via \mathcal{M} (cf. XI.4.12 [FD]). Now construct $\rho = \text{Ind}_{\mathcal{L}}^{\mathcal{F}}(\gamma) \in \text{Rep}(\mathcal{F})$ as follows. On $\mathcal{M} \otimes \mathcal{H}$ define a pre-inner product $(\cdot, \cdot)_0$ by

$$(s \otimes \xi, t \otimes \eta)_0 := (\gamma(\langle t, s \rangle)\xi, \eta) = (\gamma(t^*s)\xi, \eta) \quad (12)$$

and define from $\mathcal{M} \otimes \mathcal{H}$ the Hilbert space

$$\mathcal{K} := \overline{\mathcal{M} \otimes \mathcal{H} / \text{Ker}(\cdot, \cdot)_0}$$

where closure is obviously w.r.t. $(\cdot, \cdot)_0$. Denote the image of an elementary tensor $s \otimes \xi$ in \mathcal{K} by $s \tilde{\otimes} \xi$, and define the representation $\rho : \mathcal{F} \rightarrow \mathcal{B}(\mathcal{K})$ by

$$\rho(F)(s \tilde{\otimes} \xi) := Fs \tilde{\otimes} \xi \quad \forall s \in \mathcal{M}, \xi \in \mathcal{H}, F \in \mathcal{F}.$$

If we let $\gamma = \Lambda(\pi)$, then Equation (12) becomes

$$(s \otimes \xi, t \otimes \eta)_0 := (\tilde{\pi}(s)\xi, \tilde{\pi}(t)\eta)$$

and so we can identify \mathcal{K} with the subspace $[\tilde{\pi}(\mathcal{M})\mathcal{H}] = [\tilde{\pi}(\mathcal{FL})\mathcal{H}]$, via the unitary $U(s \otimes \xi) := \tilde{\pi}(s)\xi$. Moreover we have for all $s \tilde{\otimes} \xi, t \tilde{\otimes} \eta$ that

$$\begin{aligned} (\rho(F)(s \tilde{\otimes} \xi), t \tilde{\otimes} \eta) &= (Fs \tilde{\otimes} \xi, t \tilde{\otimes} \eta) = (\tilde{\pi}(Fs)\xi, \tilde{\pi}(t)\eta) \\ &= (\pi(F) \cdot \tilde{\pi}(s)\xi, \tilde{\pi}(t)\eta) \end{aligned}$$

and so (ρ, \mathcal{K}) is unitarily equivalent to $\pi \upharpoonright \mathcal{F}$ on $[\tilde{\pi}(\mathcal{M})\mathcal{H}]$. Now application of Lemma 1.10 (proven below) to $\tilde{\pi} \in \text{Rep} \mathcal{E}$ implies that we have $\mathcal{H} = [\tilde{\pi}(\mathcal{M})\mathcal{H}]$ iff $\pi \in \text{Rep}_{\mathfrak{S}_0} \mathcal{F}$, and the latter is what we assumed at the start. Thus ρ is unitarily equivalent to π . Furthermore, we see above that the

induction process produce representations $\tilde{\pi}$ such that $\mathcal{H} = [\tilde{\pi}(\mathcal{M})\mathcal{H}]$, hence the image under induction via \mathcal{M} of $\text{Rep } \mathcal{L}$ is $\text{Rep}_{\mathfrak{S}_0} \mathcal{F}$, using Lemma 1.10 again. ■

Lemma 1.10. *A representation $(\pi, \mathcal{H}) \in \text{Rep } \mathcal{E}$ satisfies $\mathcal{H} = [\pi(\mathcal{M})\mathcal{H}]$ iff $\pi \upharpoonright \mathcal{F}$ is normal with respect to $\pi_{\mathfrak{S}_0}(\mathcal{F})$.*

Proof: Let $\mathcal{H} = [\pi(\mathcal{M})\mathcal{H}]$ and choose a normalised vector $\psi \in \pi(\mathcal{L})\mathcal{H}$, i.e. $\psi = \pi(L)\xi$, $\|\psi\| = 1$. Let ω_ψ denote the associated vector state on \mathcal{E} , then since $\|\omega_\psi \upharpoonright \mathcal{L}\| = 1$ and \mathcal{L} is hereditary, it is everywhere determined on \mathcal{E} by its values on \mathcal{L} . Thus $\omega_\psi \upharpoonright \mathcal{F} = \theta(\omega_\psi \upharpoonright \mathcal{L}) \in \mathfrak{S}_0 \subset (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$. Since the normal functionals of any representation is a folium, hence invariant under conjugation, we conclude that also $\omega_{\pi(F)\psi} \upharpoonright \mathcal{F} = \omega_{\pi(FL)\xi} \upharpoonright \mathcal{F} \in (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$ for all $L \in \mathcal{L}$, $F \in \mathcal{F}$ and $\xi \in \mathcal{H}$, i.e. $\omega_{\pi(\mathcal{F}\mathcal{L})\mathcal{H}} \upharpoonright \mathcal{F} \in (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$. Since by assumption $\pi(\mathcal{F}\mathcal{L})\mathcal{H}$ spans a dense subspace of \mathcal{H} , it follows from Kadison and Ringrose 7.1.15 [KR] that $\pi \upharpoonright \mathcal{F}$ is normal with respect to $\pi_{\mathfrak{S}_0}(\mathcal{F})$. Conversely, let $\pi \upharpoonright \mathcal{F}$ be normal with respect to $\pi_{\mathfrak{S}_0}(\mathcal{F})$, and assume that $\mathcal{H} \neq [\pi(\mathcal{M})\mathcal{H}]$, i.e. there is some $\psi \perp [\pi(\mathcal{F}\mathcal{L})\mathcal{H}]$. First, we show that $\omega_\psi \upharpoonright \mathcal{F} \notin (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$. If not, then $\theta^{-1}(\omega_\psi \upharpoonright \mathcal{F}) \in \mathfrak{S}(\mathcal{L})$, i.e. the normal extension $\tilde{\omega}_\psi$ to \mathcal{F}'' restricts to a state on \mathcal{L} . Since this normal extension on \mathcal{E} is just $\tilde{\omega}_\psi(A) = (\psi, \pi(A)\psi)$, we conclude from the given $\psi \perp [\pi(\mathcal{F}\mathcal{L})\mathcal{H}] \supset \pi(\mathcal{L})\psi$ that $\tilde{\omega}_\psi(\mathcal{L}) = 0$, which contradicts the fact that it must be a state on \mathcal{L} . Thus $\omega_\psi \upharpoonright \mathcal{F} \notin (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$. Now we know by the first part that $\omega_{\pi(\mathcal{F}\mathcal{L})\mathcal{H}} \upharpoonright \mathcal{F} \in (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$, and in fact since the normal functionals is a norm-closed folium, we have $\omega_{[\pi(\mathcal{F}\mathcal{L})\mathcal{H}]} \upharpoonright \mathcal{F} \in (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$, so since $\mathcal{H} = \mathbb{C}\psi \oplus [\pi(\mathcal{F}\mathcal{L})\mathcal{H}]$ it is impossible to find a dense subspace $\mathcal{S} \subset \mathcal{H}$ such that $\omega_\varphi \upharpoonright \mathcal{F} \in (\pi_{\mathfrak{S}_0}(\mathcal{F}))_*$ for all $\varphi \in \mathcal{S}$, hence by Kadison and Ringrose 7.1.15 [KR] $\pi \upharpoonright \mathcal{F}$ cannot be normal w.r.t. $\pi_{\mathfrak{S}_0}(\mathcal{F})$. This contradicts our hypothesis, hence $\mathcal{H} = [\pi(\mathcal{M})\mathcal{H}]$. ■

2. Ideal hosts up to a central algebra.

Above we saw that a pair $(\mathcal{F}, \mathfrak{S}_0)$ with \mathfrak{S}_0 a folium without pure states, has no ideal host. A particularly bad case of this, is the pair $(L^\infty(X, \mu), \mathfrak{S}_N)$ where μ has no discrete part, and where \mathfrak{S}_N denotes the set of normal states, i.e. the measures absolutely continuous w.r.t. μ . In this case \mathfrak{S}_N does not even have extreme points, because for any measure ν absolutely continuous w.r.t. μ , we only need to subdivide its support to write it as a convex combination of other probability measures in this class. Only when $\text{supp}(\nu)$ is a point can we not do this, and this case does not occur since μ has no discrete part. In this section we want to argue that this example is symptomatic of the general case, in that if a pair $(\mathcal{F}, \mathfrak{S}_0)$ has no ideal host, it is because $(L^\infty(X, \mu), \mathfrak{S}_N)$ is embedded in it, and it acts as an obstruction. To be precise about the embedding, we will show that for any pair $(\mathcal{F}, \mathfrak{S}_0)$ with \mathfrak{S}_0 a folium, we can always find a quasi-host $\tilde{\mathcal{L}}$ in the sense of the next definition:

Def. Given a pair $(\mathcal{F}, \mathfrak{S}_0)$ where \mathfrak{S}_0 is a folium, a *quasi-host* for it, is a C^* -algebra $\tilde{\mathcal{L}}$ and two embeddings $\mathcal{F} \subset M(\tilde{\mathcal{L}})$ and $L^\infty(X, \mu) \subset ZM(\tilde{\mathcal{L}})$ for some measure space (X, μ) such that $\tilde{\mathfrak{S}}_\mu \upharpoonright \mathcal{F} = \mathfrak{S}_0$ where $\tilde{\mathfrak{S}}_\mu := \left\{ \omega \in \mathfrak{S}(\tilde{\mathcal{L}}) \mid \tilde{\omega} \upharpoonright L^\infty(X, \mu) \text{ is normal} \right\}$ and moreover, $\tilde{\mathfrak{S}}_\mu \upharpoonright C^*(\mathcal{F} \cup L^\infty(X, \mu))$ defines an injection for \mathfrak{S}_μ .

We will show that the given pair $(\mathcal{F}, \mathfrak{S}_0)$ has no ideal host if the measure μ is purely continuous. This is what we mean by saying $L^\infty(X, \mu)$ acts as an obstruction.

Example. Consider the von Neumann algebra $\mathcal{F} = L^\infty(X, \mu) \overline{\otimes} \mathcal{B}(\mathcal{H})$ acting on the Hilbert space $L^2(X, \mu) \otimes \mathcal{H}$, where $\overline{\otimes}$ denotes the W^* -tensor product (cf. 11.2 [KR]). Its pure states consists of product states $\omega_1 \otimes \omega_2$ such that ω_i are both pure. Thus, if we take the pair $(\mathcal{F}, \mathfrak{S}_0)$ where \mathfrak{S}_0 are the normal states of \mathcal{F} , and assume that μ has no discrete part, then \mathfrak{S}_0 has no pure states, hence this pair has no ideal host. Nevertheless, the algebra $\tilde{\mathcal{L}} = L^\infty(X, \mu) \otimes \mathcal{K}(\mathcal{H})$ is a quasi-host for $(\mathcal{F}, \mathfrak{S}_0)$.

To start the analysis, let $(\mathcal{F}, \mathfrak{S}_0)$ be a pair with \mathfrak{S}_0 a folium, then by Theorem 1.2 we construct the representation $\pi_{\mathfrak{S}_0} = \bigoplus_{\omega \in \mathfrak{S}_0} \pi_\omega$ and identify \mathfrak{S}_0 with the normal states of

the concrete algebra $\pi_{\mathfrak{S}_0}(\mathcal{F})$, hence with the normal states of the von Neumann algebra $\pi_{\mathfrak{S}_0}(\mathcal{F})''$. We first analyze a single cyclic component π_ω , $\omega \in \mathfrak{S}_0$ of the direct sum. We denote the von Neumann algebra $\pi_\omega(\mathcal{F})''$ by \mathcal{N} , and its set of normal states by \mathfrak{S}_N .

Central to the following constructions, is the usual decomposition theory with respect to some commutative subalgebra $\mathcal{C} \subset \mathcal{N}'$ (cf. Takesaki [Tak]), however, since the maps involved are only defined up to μ -negligible sets on some measure space (X, μ) , and we will actually need everywhere defined maps, we now redo some of the basic constructions in order to remedy this.

Theorem 2.1. *Let \mathcal{F} be a C^* -algebra with a fixed state $\omega \in \mathfrak{S}(\mathcal{F})$, and a commutative unital C^* -algebra $\mathcal{C} \subset \pi_\omega(\mathcal{F})'$, then the Gel'fand isomorphism $\Phi : \mathcal{C}'' \rightarrow L^\infty(X, \mu)$ equips the spectrum X of \mathcal{C} with a probability measure μ , $\text{supp } \mu = X$, and there is a map $\psi : X \rightarrow \mathfrak{S}(\mathcal{C}')$ such*

that

- (i) the map $x \rightarrow \psi_x(A)$ is in $L^\infty(X, \mu)$ for all $A \in \mathcal{C}'$, and if \mathcal{C} is a von Neumann algebra, $x \rightarrow \psi_x(A)$ is in $C(X)$ for all $A \in \mathcal{C}'$,
- (ii) $\tilde{\omega}(A) := (\Omega_\omega, A\Omega_\omega) = \int_X \psi_x(A) d\mu(x) \quad \forall A \in \mathcal{C}'$,
- (iii) $\psi_x(C \cdot A) = \Phi(C)(x) \cdot \psi_x(A) \quad \forall C \in \mathcal{C}'' , \quad A \in \mathcal{C}'$.

Proof: (This proof is based on Takesaki 6.23, p241 [Tak])

Since Ω_ω is cyclic for $\pi_\omega(\mathcal{F})$, it is separating for \mathcal{C} and \mathcal{C}'' , so $\tilde{\omega}|_{\mathcal{C}}$ is a faithful state of \mathcal{C} . Thus by the Riesz representation theorem there is a Borel measure μ on the compact set X such that

$$\tilde{\omega}(C) = \int_X \Phi(C)(x) d\mu(x), \quad C \in \mathcal{C}$$

with $\Phi : \mathcal{C} \rightarrow C(X)$ the Gel'fand isomorphism. Moreover μ is a probability measure since $\tilde{\omega}(\mathbb{I}) = 1$ and $\text{supp } \mu = X$ because $\tilde{\omega}$ is faithful. Since Ω_ω separates \mathcal{C} , we can consistently define a unitary $U : [\mathcal{C}\Omega_\omega] \rightarrow L^2(X, \mu)$ by $U(C\Omega_\omega) := \Phi(C)$ which produces the representation $\tilde{\Phi} : \mathcal{C} \rightarrow \mathcal{B}(L^2(X, \mu))$ by $\tilde{\Phi}(C) := UCU^{-1} = T_{\Phi(C)}$ in terms of multiplication operators $\{T_f \mid f \in C(X)\}$. Since unitary conjugation is a normal map, $\tilde{\Phi}$ extends to $\mathcal{C}'' \subset \pi_\omega(\mathcal{F})'$ and $\tilde{\Phi}(\mathcal{C})'' = \tilde{\Phi}(\mathcal{C}'')$, i.e. $\tilde{\Phi}(\mathcal{C}'') = \{T_f \mid f \in L^\infty(X, \mu)\}$. If \mathcal{C} is already a von Neumann algebra, this simplifies to

$$\{T_f \mid f \in C(X)\} = \tilde{\Phi}(\mathcal{C}) = \tilde{\Phi}(\mathcal{C})'' = \{T_f \mid f \in L^\infty(X, \mu)\},$$

so for each $f \in L^\infty(X, \mu)$ we can find a $\tilde{f} \in C(X)$ such that $T_f = T_{\tilde{f}}$ on $L^2(X, \mu)$, producing the isomorphism $C(X) \cong L^\infty(X, \mu)$ (this isomorphism also occurs in the proof of Theorem III.1.18 [Tak]). Henceforth we will blur the distinction between Φ and $\tilde{\Phi}$ and always make the identification with $C(X)$ if \mathcal{C} is a von Neumann algebra.

Let e be the projection of \mathcal{H}_ω onto $[\mathcal{C}\Omega_\omega] = [\mathcal{C}''\Omega_\omega]$, then $e \in \mathcal{C}'$ because \mathcal{C} preserves $[\mathcal{C}\Omega_\omega]$. Define $\Upsilon : \mathcal{C}'' \rightarrow \mathcal{B}([\mathcal{C}\Omega_\omega])$ by $\Upsilon(C) := eC$, then $\Upsilon(\mathcal{C}'') = e\mathcal{C}''$ is maximally commutative in $\mathcal{B}([\mathcal{C}\Omega_\omega])$ because it has a cyclic vector Ω_ω , cf. Takesaki Corr. 1.3, p104 [Tak]. Moreover Υ is injective because Ω_ω is separating for \mathcal{C}'' . Now $(e\mathcal{C}')' \cap [\mathcal{C}\Omega_\omega] = e\mathcal{C}'e \subset \mathcal{B}([\mathcal{C}\Omega_\omega])$ (easily verified, but also in Takesaki 3.10 [Tak]), hence $e\mathcal{C}'e \subset e\mathcal{C}''$ because the latter is maximally commutative in $\mathcal{B}([\mathcal{C}\Omega_\omega])$ (also by Takesaki 3.10). Define $\delta : \mathcal{C}' \rightarrow \Upsilon(\mathcal{C}'')$ by $\delta(A) := eAe \in e\mathcal{C}''$. Clearly $\delta(\mathbb{I}) = e$ and δ is positive and extends Υ . Next define $\psi : X \rightarrow \mathfrak{S}(\mathcal{C}')$ by

$$\psi_x(A) := \Phi(\Upsilon^{-1}(\delta(A)))(x)$$

and note $\Phi(\Upsilon^{-1}(\delta(A)))$ is in $L^\infty(X, \mu)$, so ψ is defined μ -almost everywhere, and if \mathcal{C} is a von Neumann algebra we can identify $\Phi(\Upsilon^{-1}(\delta(A)))$ with

an element of $C(X)$ in which case ψ is everywhere defined. By positivity and normalisation of the various maps each ψ_x is a state. Now

$$\begin{aligned}\tilde{\omega}(\Upsilon^{-1}(\delta(A))) &= \int_X \Phi(\Upsilon^{-1}(\delta(A)))(x) d\mu(x) \\ &= \int_X \psi_x(A) d\mu(x) \\ \tilde{\omega}(A) &= (\Omega_\omega, A\Omega_\omega) = (\Omega_\omega, eAe\Omega_\omega) \quad \text{using } e\Omega_\omega = \Omega_\omega \\ &= (\Omega_\omega, \delta(A)\Omega_\omega) = (\Omega_\omega, \Upsilon^{-1}(\delta(A))\Omega_\omega) \\ &= \tilde{\omega}(\Upsilon^{-1}(\delta(A))) = \int_X \psi_x(A) d\mu(x) .\end{aligned}$$

For (ii), observe that for $C \in \mathcal{C}''$, $A \in \mathcal{C}'$ we have

$$\psi_x(C \cdot A) = \Phi(\Upsilon^{-1}(\delta(CA)))(x) = \Phi(C)(x) \cdot \psi_x(A)$$

simply using the fact that Φ, Υ^{-1} and δ are homomorphisms and by definition $\Upsilon^{-1}(\delta(C)) = C$ for $C \in \mathcal{C}''$. ■

- Remarks.** (1) A very important point here is that for \mathcal{C} a von Neumann algebra, $\psi : X \rightarrow \mathfrak{S}(\mathcal{C}')$ is everywhere defined. This comes from the identification which Φ makes between $C(X)$ and $L^\infty(X, \mu)$. The spectrum X of \mathcal{C} in this case is of course hyperstonean hence extremely disconnected (cf. [Tak]).
- (2) Note that no separability assumption was needed here.
- (3) The map ψ and measure μ produces precisely the orthogonal barycentric decomposition of $\tilde{\omega}$ used in decomposition theory in e.g. Takesaki [Tak] or Bratteli and Robinson [BR]. One uses ψ to write μ as a measure on the state space $\mathfrak{S}(\mathcal{F})$.
- (4) By restricting $\tilde{\omega}$ and ψ_x to any subalgebra of \mathcal{C}' (e.g. $\pi_\omega(\mathcal{F})$ or $\pi_\omega(\mathcal{F})''$), we obtain a decomposition theorem for states on these. Note that the usual theory assumes that \mathcal{C} is a von Neumann algebra.

Def. Given the data $\mathcal{F}, \omega, \mathcal{C}$ of theorem 2.1 assume that \mathcal{C} is a von Neumann algebra with subsequent map $\psi : X \rightarrow \mathfrak{S}(\mathcal{C}')$ and define the following two bundles on X :

- $\mathcal{H}(\psi)$ is the bundle with projection $p : \mathcal{H}(\psi) \rightarrow X$ by $p^{-1}(x) = \mathcal{H}_{\psi_x}$,
- $\mathcal{B}(\psi)$ the bundle with $q : \mathcal{B}(\psi) \rightarrow X$ by $q^{-1}(x) = \mathcal{B}(\mathcal{H}_{\psi_x})$.

At this point there is no topology, so the total spaces of these bundles are just the unions of their fibres. Clearly the sections $\Gamma(\mathcal{B}(\psi))$ act pointwise on the sections $\Gamma(\mathcal{H}(\psi))$.

Now there are some canonical families of sections:

- $\Omega \in \Gamma(\mathcal{H}(\psi))$ is the section $\Omega(x) := \Omega_{\psi_x}$,
- $\Pi(A) \in \Gamma(\mathcal{B}(\psi))$ is the section $\Pi(A)(x) := \pi_{\psi_x}(A)$, $A \in \mathcal{C}'$.

Clearly the latter specialises on \mathcal{C} to

- $\Pi(C)(x)\varphi = \Phi(C)(x)\varphi \quad \forall \varphi \in \mathcal{H}_{\psi_x}, C \in \mathcal{C}$.

Let $c = \Pi(C)\Omega$, $d = \Pi(D)\Omega$ with $C, D \in \mathcal{C}'$, then

$$(c(x), d(x))_{\mathcal{H}_{\psi_x}} = (\Pi(C)\Omega(x), \Pi(D)\Omega(x))_{\mathcal{H}_{\psi_x}} = \psi_x(C^*D)$$

which is a continuous function in x by theorem 2.1. Obviously $\Pi(\mathcal{C}')\Omega$ is a linear space, and we have that $x \rightarrow (c(x), d(x))_{\mathcal{H}_{\psi_x}}$ is continuous, hence integrable (since X compact, μ a probability measure). Thus we can equip $\Pi(\mathcal{C}')\Omega$ with the inner product

$$(c, d) := \int_X (c(x), d(x))_{\mathcal{H}_{\psi_x}} d\mu(x) .$$

and hence form the Hilbert space

$$\mathcal{H}_\Gamma := \overline{\Pi(\mathcal{C}')\Omega / \text{Ker}(\cdot, \cdot)}$$

with factorisation map $\kappa : \Pi(\mathcal{C}')\Omega \rightarrow \mathcal{H}_\Gamma$.

Theorem 2.2. $\Pi(\mathcal{C}')$ lifts through κ to define a representation $\pi : \mathcal{C}' \rightarrow \mathcal{B}(\mathcal{H}_\Gamma)$. Then $\kappa(\Omega)$ is cyclic for $\pi(\mathcal{F})$, and there is a unitary $U : \mathcal{H}_\omega \rightarrow \mathcal{H}_\Gamma$ which intertwines the representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ of \mathcal{C}' with $(\pi, \mathcal{H}_\Gamma, \kappa(\Omega))$.

Proof: Obviously $\Pi(\mathcal{C}')$ preserves $\Pi(\mathcal{C}')\Omega$; we show that it preserves $\text{Ker}(\cdot, \cdot)$. Let $\varphi = \Pi(A)\Omega \in \text{Ker}(\cdot, \cdot)$, i.e.

$$0 = \|\varphi\|^2 = \int_X (\Omega(x), \Pi(A^*A)(x)\Omega(x)) d\mu(x) = \int_X \psi_x(A^*A) d\mu(x)$$

Thus $\psi_x(A^*A) = 0$ by positivity and continuity of this map. Let $B \in \mathcal{C}'$, then

$$\|\Pi(B)\varphi\|^2 = \int_X \psi_x(A^*B^*BA) d\mu(x) .$$

By the Cauchy–Schwartz inequality $\psi_x(A^*B^*BA) = 0$ using $\psi_x(A^*A) = 0$. Thus $\Pi(B)$ preserves $\text{Ker}(\cdot, \cdot)$ and hence lifts through κ . To show that $\kappa(\Omega)$ is cyclic for $\Pi(\mathcal{F})$, assume the converse, i.e. there is a sequence $\{\varphi_n\} \subset \Pi(\mathcal{C}')\Omega$ converging with respect to the seminorm $\|\varphi\| := [\int_X \|\varphi(x)\|_{\mathcal{H}_{\psi_x}}^2 d\mu(x)]^{1/2}$ and $(\varphi_n, \Pi(\mathcal{F})\Omega) \rightarrow 0$ as $n \rightarrow \infty$, but $\|\varphi_n\|$ does not converge to zero. (This means there is some nonzero $\tilde{\varphi} \in \mathcal{H}_\Gamma$ which is orthogonal to $\Pi(\mathcal{F})\Omega$). Let $\varphi_n = \Pi(A_n)\Omega$, $A_n \in \mathcal{C}'$, then

$$\begin{aligned} (\varphi_n, \Pi(B)\Omega) &= \int_X \psi_x(A_n^*B) d\mu(x) = (\Omega_\omega, A_n^*B\Omega_\omega) \\ &= (A_n\Omega_\omega, B\Omega_\omega) \rightarrow 0 \quad \forall B \in \pi_\omega(\mathcal{F}). \end{aligned}$$

However Ω_ω is cyclic for $\pi_\omega(\mathcal{F})$ in \mathcal{H}_ω , hence $A_n\Omega_\omega \rightarrow 0$ and so since $\|\varphi_n\|^2 = \|A_n\Omega_\omega\|^2 \rightarrow 0$, we have contradicted the hypothesis, so $\kappa(\Omega)$ must be cyclic for $\Pi(\mathcal{F})$. Now since

$$(\Omega_\omega, A\Omega_\omega) = \tilde{\omega}(A) = \int_X \psi_x(A) d\mu(x) = (\Omega, \Pi(A)\Omega)$$

the unitary equivalence follows from the GNS-theorem. ■

Remarks. (i) Since $(\pi_{\tilde{\omega}}, \mathcal{H}_{\tilde{\omega}}, \Omega_{\tilde{\omega}})$ is the representation in which \mathcal{C}' is defined, Π is an isomorphism.

(ii) We can realise vectors $\xi \in \mathcal{H}_{\Gamma}$ as sections $x \rightarrow \xi(x) \in \mathcal{H}_{\psi_x}$ (almost everywhere) such that

$$(\xi, \phi) = \int_X (\xi(x), \phi(x))_{\mathcal{H}_{\psi_x}} d\mu(x),$$

but we will not need it here, so omit it.

Def. Let $\varphi, \zeta \in \Pi(\mathcal{C}')\Omega \subset \Gamma(\mathcal{H}(\psi))$ and define a section $\tilde{A}_{\varphi, \zeta} \in \Gamma(\mathcal{B}(\psi))$ by

$$(\tilde{A}_{\varphi, \zeta} c)(x) := \varphi(x) (\zeta(x), c(x))_{\mathcal{H}_{\psi_x}} \quad \forall c \in \Gamma(\mathcal{H}(\psi)).$$

Lemma 2.3. $\tilde{A}_{\varphi, \zeta}$ preserves $\Pi(\mathcal{C}')\Omega$ and $\text{Ker}(\cdot, \cdot)$, and is bounded on these spaces, hence defines an operator $A_{\varphi, \zeta} \in \mathcal{B}(\mathcal{H}_{\Gamma})$ by $A_{\varphi, \zeta} \kappa(\eta) := \kappa(\tilde{A}_{\varphi, \zeta} \eta)$ for $\eta \in \Pi(\mathcal{C}')\Omega$.

Proof: Let $\varphi = \Pi(A)\Omega$, $\zeta = \Pi(B)\Omega$, $\eta = \Pi(C)\Omega$, $A, B, C \in \mathcal{C}'$, then

$$\begin{aligned} (\tilde{A}_{\varphi, \zeta} \eta)(x) &= \pi_{\psi_x}(A)\Omega_{\psi_x} \cdot (\pi_{\psi_x}(B)\Omega_{\psi_x}, \pi_{\psi_x}(C)\Omega_{\psi_x}) \\ &= \psi_x(B^*C) \cdot \pi_{\psi_x}(A)\Omega_{\psi_x} = \Phi(\Upsilon^{-1}(\delta(B^*C)))(x) \cdot \Pi(A)\Omega(x) \\ &= \Pi(\Upsilon^{-1}(\delta(B^*C))A)\Omega(x) \end{aligned}$$

where we used the fact that $\Upsilon^{-1}(\delta(B^*C)) \in \mathcal{C}'' = \mathcal{C}$ and theorem 2.1. Thus $\tilde{A}_{\varphi, \zeta} \eta \in \Pi(\mathcal{C}')\Omega$. That $\tilde{A}_{\varphi, \zeta}$ preserves $\text{Ker}(\cdot, \cdot)$ will follow from the next calculation as well as the remaining claims. Since by theorem 2.2, $\kappa(\Omega)$ is cyclic for $\pi(\mathcal{F})$, it suffices to do the calculation on $\Pi(\mathcal{F})\Omega$, so now let $\eta = \Pi(C)\Omega$, $C \in \pi_{\omega}(\mathcal{F})$. Then

$$\begin{aligned} \|\tilde{A}_{\varphi, \zeta} \eta\|^2 &= \int |\psi_x(B^*C)|^2 \psi_x(A^*A) d\mu(x) \\ &\leq \int_X \psi_x(B^*B) \cdot \psi_x(C^*C) \cdot \psi_x(A^*A) d\mu(x) \\ &\leq \|A\|^2 \|B\|^2 \int_X \psi_x(C^*C) d\mu(x) \\ &= \|A\|^2 \|B\|^2 \|\eta\|^2. \end{aligned}$$

Define the norm $\|a\| = \sup_{x \in X} \|a(x)\|_{\mathcal{B}(\mathcal{H}_{\psi_x})}$ on the space of bounded sections:

$\Gamma_0(\mathcal{B}(\psi)) := \{a \in \Gamma(\mathcal{B}(\psi)) \mid \|a\| < \infty\}$ which makes it into a C^* -algebra.

Lemma 2.4. $\{\tilde{A}_{\varphi, \zeta} \mid \varphi, \zeta \in \Pi(\mathcal{C}')\Omega\} \subset \Gamma_0(\mathcal{B}(\psi))$.

Proof: $(\tilde{A}_{\varphi, \zeta} c)(x) = \varphi(x) (\zeta(x), c(x))$ for $c \in \Gamma(\mathcal{H}(\psi))$, hence

$$\begin{aligned} \left\| (\tilde{A}_{\varphi, \zeta} c)(x) \right\| &\leq \|\varphi(x)\| \cdot \|\zeta(x)\| \cdot \|c(x)\|, \quad \text{and} \\ \left\| \tilde{A}_{\varphi, \zeta}(x) \right\|_{\mathcal{B}(\mathcal{H}_{\psi_x})} &= \|\varphi(x)\| \cdot \|\zeta(x)\|, \quad \text{so} \\ \left\| \tilde{A}_{\varphi, \zeta} \right\| &= \sup_{x \in X} (\|\varphi(x)\| \cdot \|\zeta(x)\|) = \sup_{x \in X} \left(\psi_x(A^* A) \psi_x(B^* B) \right)^{1/2} \\ &\leq \|A\| \cdot \|B\| < \infty \end{aligned}$$

where we assumed $\varphi = \Pi(A)\Omega$, $\zeta = \Pi(B)\Omega$. ■

Using the C^* -operations of $\Gamma_0(\mathcal{B}(\psi))$, we now define

$$\tilde{\mathcal{L}} := C^* \left\{ \tilde{A}_{\varphi, \zeta} \mid \varphi, \zeta \in \Pi(\mathcal{C}')\Omega \right\} \subset \Gamma_0(\mathcal{B}(\psi)).$$

Theorem 2.5. *With the data \mathcal{F} , ω , $\mathcal{C} = \mathcal{C}''$ above;—*

- (i) $\tilde{\mathcal{L}}(x) := \{ a(x) \mid a \in \mathcal{L} \} = \mathcal{K}(\mathcal{H}_{\psi_x})$
- (ii) $\Pi(\mathcal{C}')\tilde{\mathcal{L}} \subset \tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}\Pi(\mathcal{C}')$ and $[\tilde{\mathcal{L}}, \Pi(\mathcal{C})] = 0$,
- (iii) $\tilde{\mathcal{L}}$ lifts through κ to define a representation $\rho : \tilde{\mathcal{L}} \rightarrow \mathcal{B}(\mathcal{H}_\Gamma)$. Thus by (ii) $\rho(\tilde{\mathcal{L}}) \subset \pi(\mathcal{C}')$.

Proof: (i) In a fibre we have $\left\| \tilde{A}_{\varphi, \zeta}(x) \right\|_{\mathcal{B}(\mathcal{H}_{\psi_x})} = \|\varphi(x)\| \cdot \|\zeta(x)\|$ so $\tilde{A}_{\varphi, \zeta}(x)$ is continuous in $\varphi(x)$, $\zeta(x)$, and thus the norm closure of $\left\{ \tilde{A}_{\varphi, \zeta}(x) \mid \varphi, \zeta \in \Pi(\mathcal{C}')\Omega \right\}$ contains all rank one operators, using the fact that $(\Pi(\mathcal{C}')\Omega)(x)$ is dense in \mathcal{H}_{ψ_x} . Since $\mathcal{K}(\mathcal{H}_{\psi_x})$ is spanned by its rank one operators and closure in a supremum norm produces pointwise closure, we get that $\tilde{\mathcal{L}}(x) = \mathcal{K}(\mathcal{H}_{\psi_x})$.
(ii) Let $\varphi = \Pi(A)\Omega$, $\zeta = \Pi(B)\Omega$, then for $E \in \mathcal{C}'$:

$$\begin{aligned} (\pi(E)\tilde{A}_{\varphi, \zeta} c)(x) &= \pi_{\psi_x}(EA)\Omega_{\psi_x} \cdot (\pi_{\psi_x}(B)\Omega_{\psi_x}, c(x)) \\ &= (\tilde{A}_{\xi, \zeta} c)(x) \quad \text{where } \xi := \Pi(EA)\Omega. \end{aligned}$$

So $\Pi(E)\tilde{A}_{\varphi, \zeta} \in \tilde{\mathcal{L}}$. Similarly $\tilde{A}_{\varphi, \zeta}\Pi(E) \in \tilde{\mathcal{L}}$, and as $\tilde{A}_{\varphi, \zeta}(x)^* = \tilde{A}_{\zeta, \varphi}(x)$ it follows that $\Pi(\mathcal{C}')\tilde{\mathcal{L}} \subset \tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}\Pi(\mathcal{C}')$. To see that $[\tilde{\mathcal{L}}, \Pi(\mathcal{C})] = 0$, let $F \in \mathcal{C}$, so

$$\begin{aligned} (\tilde{A}_{\varphi, \zeta}\Pi(F)c)(x) &= \varphi(x) (\zeta(x), \Pi(F)c(x)) = \varphi(x) (\zeta(x), \Phi(F)(x)c(x)) \\ &= \Phi(F)(x) (\tilde{A}_{\varphi, \zeta} c)(x) = (\Pi(F)\tilde{A}_{\varphi, \zeta} c)(x). \end{aligned}$$

(iii) By lemma 2.3 we already know that $\tilde{A}_{\varphi, \zeta}$ lifts through κ , so since lifting is a homomorphism, we define $\rho(\tilde{A}_{\varphi, \zeta}) := \tilde{A}_{\varphi, \zeta} \in \mathcal{B}(\mathcal{H}_\Gamma)$ and check continuity. Let $\varphi = \Pi(A)\Omega$, $\zeta = \Pi(B)\Omega$, $\eta = \Pi(\mathcal{C}')\Omega$ then

$$\left\| \rho(\tilde{A}_{\varphi, \zeta})\kappa(\eta) \right\|^2 = \left\| \kappa(\tilde{A}_{\varphi, \zeta}\eta) \right\|^2 = \int_X \left\| (\tilde{A}_{\varphi, \zeta}\eta)(x) \right\|^2 d\mu(x)$$

$$\begin{aligned}
&= \int_X |\psi_x(B^*C)|^2 \cdot \psi_x(A^*A) d\mu(x) \\
&\leq \int_X \psi_x(B^*B) \cdot \psi_x(C^*C) \cdot \psi_x(A^*A) d\mu(x) \\
&\leq \sup_{x \in X} (\psi_x(A^*A) \cdot \psi_x(B^*B)) \int_X \psi_x(C^*C) d\mu(x) \\
&= \|\tilde{A}_{\varphi, \zeta}\|^2 \cdot \|\kappa(\eta)\|^2
\end{aligned}$$

■

Remarks. (1) Note that theorem 2.5(ii) expresses that $\tilde{\mathcal{L}}$ is a $C(X)$ -algebra, given that $\Pi(\mathcal{C}) = C(X)$. Thus the family $\{\tilde{\mathcal{L}}(x) \mid x \in X\}$ can be topologised as a Fell bundle (cf. M. Nilsen [Ni]). In fact, since \mathcal{C}' is also obviously a $C(X)$ -algebra, it is also the set of continuous sections of a Fell bundle. Note also that $\tilde{\mathcal{L}}$ has ideals corresponding to closed subsets Y of X by $\mathcal{I}_Y := \{L \in \mathcal{L} \mid \Pi(L)|_Y = 0\}$. Obviously, since $\tilde{\mathcal{L}}$ is fibrewise the compacts, it is a nonunital algebra.

(2) Another, possibly smaller choice for $\tilde{\mathcal{L}}$ is

$$\mathcal{L}_{\mathcal{F}} := C^* \left\{ \tilde{A}_{\varphi, \zeta} \mid \varphi, \zeta \in \text{Span}\{\Pi(\mathcal{C}\pi_{\omega}(\mathcal{F})'')\Omega\} \right\}$$

in which case we still have fibrewise the compacts, $\mathcal{L}_{\mathcal{F}}(x) = \mathcal{K}(\mathcal{H}_{\psi_x})$, but instead of 2.4(ii) we now have only the weaker property that $\Pi(\tilde{\mathcal{C}})$ and $\Pi(\pi_{\omega}(\mathcal{F})'')$ are in the relative multiplier algebra of $\mathcal{L}_{\mathcal{F}}$ in $\Gamma_0(\mathcal{B}(\psi))$.

Lemma 2.6. (i) Let $a \in \tilde{\mathcal{L}}$, then the map $x \rightarrow \|a(x)\|$ is continuous.
(ii) Fix an $x \in X$, then $\{a \in \tilde{\mathcal{L}} \mid a(x) = 0\}$ is dense in the set $\{f \cdot a \mid a \in \tilde{\mathcal{L}}, f \in C(X), f(x) = 0\}$.

Proof: (i) First we show that $x \rightarrow \|a(x)\|$ is continuous for the generating set of $\tilde{\mathcal{L}}$. Let $a = \tilde{A}_{\varphi, \zeta}$ with $\varphi = \Pi(A)\Omega$, $\zeta = \Pi(B)\Omega$, $A, B \in \mathcal{C}'$, then

$$\begin{aligned}
\|a(x)\| &= \|\tilde{A}_{\varphi, \zeta}(x)\| = \|\varphi(x)\| \cdot \|\zeta(x)\| \\
&= \|\pi_{\psi_x}(A)\Omega_{\psi_x}\| \cdot \|\pi_{\psi_x}(B)\Omega_{\psi_x}\| = [\psi_x(A^*A) \psi_x(B^*B)]^{1/2}
\end{aligned}$$

which is continuous in x because $\psi_x(A^*A) = \Phi(\Upsilon^{-1}(\delta(A^*A)))(x)$ and this is continuous by theorem 2.1(i). It is easy to see that $x \rightarrow \|a(x)\|$ is continuous for linear combinations of the $\tilde{A}_{\varphi, \zeta}$: let $a, b \in \Gamma(\mathcal{B}(\psi))$ be such that $x \rightarrow \|a(x)\|$ and $x \rightarrow \|b(x)\|$ is continuous, then for a convergent net $x_{\nu} \rightarrow x$ in X ,

$$\begin{aligned}
\|a(x_{\nu}) + b(x_{\nu})\| - \|a(x) + b(x)\| &\leq \|a(x_{\nu}) + b(x_{\nu}) - a(x) - b(x)\| \\
&\leq \|a(x_{\nu}) - a(x)\| + \|b(x_{\nu}) - b(x)\| \longrightarrow 0.
\end{aligned}$$

Next we show that $\mathcal{Y} := \text{Span} \left\{ \tilde{A}_{\varphi, \zeta} \mid \varphi, \zeta \in \Pi(\mathcal{F})\Omega \right\}$ is in fact a dense *-subalgebra of $\tilde{\mathcal{L}}$, not just a generating set. That it is involutive follows from $\tilde{A}_{\varphi, \zeta}^* = \tilde{A}_{\zeta, \varphi}$. We only need to show that $\tilde{A}_{\varphi, \zeta} \cdot \tilde{A}_{\xi, \eta} \in \mathcal{Y}$. Let $c \in \Gamma(\mathcal{H}(\psi))$, then

$$\begin{aligned} (\tilde{A}_{\varphi, \zeta} \cdot \tilde{A}_{\xi, \eta} c)(x) &= \varphi(x) (\zeta(x), (\tilde{A}_{\xi, \eta} c)(x)) \\ &= \varphi(x) (\zeta(x), \xi(x)) \cdot (\eta(x), c(x)) = (\tilde{A}_{\bar{\varphi}, \eta} c)(x) \end{aligned}$$

where $\bar{\varphi} := \Pi(\widetilde{(\zeta, \xi)})\varphi \in \Pi(\mathcal{C}')\Omega$, since $x \rightarrow \widetilde{(\zeta, \xi)}(x) := (\zeta(x), \xi(x))$ is continuous so corresponds to an element $\widetilde{(\zeta, \xi)}$ of \mathcal{C} . Thus

$$\tilde{A}_{\varphi, \zeta} \cdot \tilde{A}_{\xi, \eta} \in \mathcal{Y}.$$

Finally, we need to show that if $\{a_n\} \subset \mathcal{Y}$ is a sequence converging to an $a \in \tilde{\mathcal{L}}$ in norm, then $x \rightarrow \|a(x)\|$ is continuous. Given a converging net $x_\nu \rightarrow x$ in X ,

$$\begin{aligned} \left| \|a(x_\nu)\| - \|a(x)\| \right| &\leq \|a(x_\nu) - a(x)\| \\ &= \|a(x_\nu) - a_n(x_\nu) + a_n(x_\nu) - a_n(x) + a_n(x) - a(x)\| \\ &\leq \|a(x_\nu) - a_n(x_\nu)\| + \|a_n(x_\nu) - a_n(x)\| + \|a_n(x) - a(x)\| \\ &\leq 2\|a - a_n\| + \|a_n(x_\nu) - a_n(x)\| \end{aligned}$$

and this can be made arbitrary small for suitable choices of n and ν .

(ii) Denote $\mathcal{I}_x := \left\{ a \in \tilde{\mathcal{L}} \mid a(x) = 0 \right\}$ and $\mathcal{J}_x :=$

$\overline{\left\{ f\tilde{\mathcal{L}} \mid f \in C(X), f(x) = 0 \right\}}$. If $g = f \cdot a \in \mathcal{J}_x$, we clearly have that $g(x) = f(x)a(x) = 0$, i.e. $\mathcal{J}_x \subseteq \mathcal{I}_x$. Conversely, let $a \in \mathcal{I}_x$, then by part (i) of this lemma, $U_\varepsilon := \{y \in X \mid \|a(y)\| < \varepsilon\}$ is open for any $\varepsilon > 0$. Since X is the spectrum of a von Neumann algebra, it is completely regular, so there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$, $f(y) = 1$ for all $y \notin U_\varepsilon$. So

$$\|(a - fa)(x)\| = \|a(x)(1 - f(x))\| = \|a(x)\| \cdot |1 - f(x)| = 0$$

as $a(x) = 0$. Since $\|(a - fa)(y)\| < \varepsilon$ for all $y \in U_\varepsilon$ and $\|(a - fa)(y)\| = 0$ when $y \notin U_\varepsilon$, it is clear that $\mathcal{I}_x = \mathcal{J}_x$. ■

Lemma 2.7. Any pure state γ of $\tilde{\mathcal{L}}$ is of the form $\gamma(a) = \gamma_x(a(x))$ for all $a \in \tilde{\mathcal{L}}$ where x is a distinguished point $x \in X$, and γ_x is a pure state of $\tilde{\mathcal{L}}(x) = \mathcal{K}(\mathcal{H}_{\psi_x})$.

Proof: From 2.5 we have $C(X)\tilde{\mathcal{L}} \subset \tilde{\mathcal{L}} \subset \tilde{\mathcal{L}}C(X)$ using $\Pi(\mathcal{C}) = C(X)$ in the explicit action, cf. the definition below 2.1. Hence by Dixmier 2.11.7 [Di] there is a unique extension $\tilde{\gamma}$ of γ to a pure state of $C^*(\tilde{\mathcal{L}} \cup \Pi(\mathcal{C})) \subset M(\tilde{\mathcal{L}})$. Since $C(X) = \Pi(\mathcal{C}) \subset Z(C^*(\tilde{\mathcal{L}} \cup \Pi(\mathcal{C})))$, $\tilde{\gamma}|_{C(X)}$ is

pure (since the image of the centre of a C*-algebra under an irreducible representation is one-dimensional). Thus $\tilde{\gamma}|_{C(X)}$ is evaluation at some distinguished point $x \in X$, and obviously the left kernel

$$N_{\tilde{\gamma}|_{C(X)}} = \text{Ker } \tilde{\gamma}|_{C(X)} = \{ f \in C(X) \mid f(x) = 0 \} .$$

Since $C(X)$ commutes with $\tilde{\mathcal{L}}$, this means

$$|\gamma(Lf)|^2 \leq \gamma(L^*L) \gamma(f^*f) = 0 \quad \forall f \in \text{Ker } \tilde{\gamma}|_{C(X)}, \quad L \in \tilde{\mathcal{L}}$$

i.e. $\{ \tilde{\mathcal{L}}f \mid f(x) = 0 \} \subset N_\gamma$. Since γ is pure,

$$\text{Ker } \gamma = N_\gamma + N_\gamma^* \supset \overline{\{ \tilde{\mathcal{L}}f \mid f(x) = 0 \}}$$

using Dixmier 2.9.1 [Di]. The last set is a two-sided ideal, hence in $\text{Ker } \pi_\gamma$, and moreover by lemma 2.6(ii), $\tilde{\mathcal{L}}(x) = \tilde{\mathcal{L}} / \overline{\{ \tilde{\mathcal{L}}f \mid f(x) = 0 \}}$ and so

$$\gamma(a) = \gamma(a + \overline{\{ \tilde{\mathcal{L}}f \mid f(x) = 0 \}}) = \hat{\gamma}(a(x))$$

defines a state $\hat{\gamma}$ on $\tilde{\mathcal{L}}(x)$, henceforth denoted by γ_x , i.e. $\gamma(a) = \gamma_x(a(x))$. Clearly if γ_x is not pure, we can write it as a convex combination of states, which through the last expression produces a convex combination for γ , contradicting the fact that γ is pure. Thus γ_x must also be pure. ■

- Remarks.** (1) Given that $\tilde{\mathcal{L}}$ can be realised as the continuous sections of a Fell bundle, lemma 2.7 is well-known, cf. e.g. Fell and Doran Prop. 8.8 p582 [FD].
- (2) C*-algebras of the form of $\tilde{\mathcal{L}}$ are well-studied in the literature as fields of elementary algebras, cf. Dixmier [Di].

Theorem 2.8. *Let γ be a state on $\tilde{\mathcal{L}}$, then there is a probability measure ν on X and a ν -almost everywhere defined map $\rho : \text{supp } \nu \rightarrow \prod_{x \in X} \mathfrak{S}(\tilde{\mathcal{L}}(x))$ such that $\rho_x \in \mathfrak{S}(\tilde{\mathcal{L}}(x))$, and*

$$\gamma(a) = \int_X \rho_x(a(x)) d\nu(x) \quad \forall a \in \tilde{\mathcal{L}} .$$

Proof: Since by 2.5 we have $\Pi(C')$ in $M(\tilde{\mathcal{L}})$, both γ and π_γ extend uniquely (on the same space) to it. Consider the unital C*-algebra $\tilde{\mathcal{C}} := \pi_\gamma(\Pi(C)) \subset \pi_\gamma(\tilde{\mathcal{L}})'$. It will only be a von Neumann algebra if γ is normal on $\Pi(C)$, which we cannot assume. Nevertheless, we can apply theorem 2.1 (which works for also for a commutative C*-algebra in the commutant) to the triple $\mathcal{L}, \gamma, \tilde{\mathcal{C}}$ to obtain a probability measure $\tilde{\nu}$ on the spectrum Y of $\tilde{\mathcal{C}}$ with support Y ,

and the Gel'fand isomorphism $\tilde{\Phi} : \tilde{\mathcal{C}} \rightarrow L^\infty(Y, \tilde{\nu})$ and a $\tilde{\nu}$ -a.e. defined map $\tilde{\psi} : Y \rightarrow \mathfrak{S}(\tilde{\mathcal{L}})$ such that

$$\gamma(a) = \int_Y \tilde{\psi}_x(a) d\tilde{\nu}(x) \quad \text{and} \quad \tilde{\psi}_x(C.a) = \tilde{\Phi}(C)(x) \cdot \tilde{\psi}_x(a)$$

for all $a \in \tilde{\mathcal{L}}$, $C \in \tilde{\mathcal{C}}$. Now $\tilde{\mathcal{C}}$ is a homomorphic image of $\mathcal{C} \cong C(X)$, and as all ideals in $C(X)$ are of the form $\{f \in C(X) \mid f(Z) = 0\}$ for some closed set $Z \subset X$, the homomorphic images of $C(X)$ are all isomorphic to the algebras $C(\overline{X \setminus Z}) = C(V)$ for $V \subset X$ the closure of an open set. Thus there is a homeomorphism $\beta : Y \rightarrow V \subset X$, with V the closure of an open set and we obtain it as follows. Since π_γ must map maximal ideals of \mathcal{C} to maximal ideals of $\tilde{\mathcal{C}}$, define $\beta : Y \rightarrow X$ by $\pi_\gamma(\mathcal{I}_{\beta(x)}) = \tilde{\mathcal{I}}_x$ where $\mathcal{I}_x := \{C \in \mathcal{C} \mid \Phi(C)(x) = 0\}$ and $\tilde{\mathcal{I}}_x := \{D \in \tilde{\mathcal{C}} \mid \tilde{\Phi}(D)(x) = 0\}$. Observe that this implies

$$\tilde{\Phi}(\pi_\gamma(C))(x) = \Phi(C)(\beta(x)).$$

We can thus write the integral of γ over X instead of Y ; - define a measure ν on X by setting it equal to $\nu = \tilde{\nu} \circ \beta^{-1}$ on $\beta(Y) \subset X$, and zero outside of this set. Then $\gamma(a) = \int_X \tilde{\psi}_{\beta^{-1}(x)}(a) d\nu(x)$ for $a \in \tilde{\mathcal{L}}$. In order to define the map ρ_x from this, we need to show for ν -almost all x that $\tilde{\psi}_{\beta^{-1}(x)}(a)$ only depends on the value $a(x)$ of $a \in \tilde{\mathcal{L}}$. Now

$$\tilde{\psi}_{\beta^{-1}(x)}(C.a) = \tilde{\Phi}(C)(\beta^{-1}(x)) \cdot \tilde{\psi}_{\beta^{-1}(x)}(a)$$

so if $C \in \tilde{\mathcal{I}}_{\beta^{-1}(x)} = \pi_\gamma(\mathcal{I}_x)$, then $\tilde{\psi}_{\beta^{-1}(x)}(C.a) = 0$ for all $a \in \tilde{\mathcal{L}}$. But by lemma 2.6(ii),

$$\overline{\mathcal{I}_x \cdot \tilde{\mathcal{L}}} = \{a \in \tilde{\mathcal{L}} \mid a(x) = 0\} =: K_x$$

so $\tilde{\psi}_{\beta^{-1}(x)}(K_x) = 0$, and thus using $\tilde{\mathcal{L}}(x) = \tilde{\mathcal{L}}/K_x$, we have

$$\tilde{\psi}_{\beta^{-1}(x)}(a) = \tilde{\psi}_{\beta^{-1}(x)}(a + K_x) =: \rho_x(a(x)), \quad a \in \tilde{\mathcal{L}}$$

and obviously $\rho_x \in \mathfrak{S}(\tilde{\mathcal{L}}(x))$. Thus finally, $\gamma(a) = \int_X \rho_x(a(x)) d\nu(x)$ for $a \in \tilde{\mathcal{L}}$. ■

Corollary 2.9. *With the data \mathcal{F} , ω , \mathcal{C} above,*

- (i) *for any state γ of $\tilde{\mathcal{L}}$ there is a probability measure ν on X and a ν -almost everywhere defined map $T : \text{supp } \nu \rightarrow \prod_{x \in X} \mathcal{K}(\mathcal{H}_{\psi_x})$ such that $T_x \in \mathcal{K}(\mathcal{H}_{\psi_x})$ is positive, trace class, normalised, and satisfies*

$$\gamma(a) = \int_X \text{Tr}(a(x) T_x) d\nu(x), \quad \forall a \in \tilde{\mathcal{L}}.$$

(ii) Let $\theta(\gamma)$ denote the unique extension of a state $\gamma \in \mathfrak{S}(\tilde{\mathcal{L}})$ to \mathcal{C}' , then

$$\theta(\gamma)(A) = \int_X \text{Tr}(\pi_{\psi_x}(A) T_x) d\nu(x), \quad \forall A \in \mathcal{C}'$$

where ν and T are as above.

Proof: (i) This follows directly from theorem 2.8 and the fact that $\tilde{\mathcal{L}}(x) = \mathcal{K}(\mathcal{H}_{\psi_x})$ for all $x \in X$. So, since all states on the compacts are of the form $\phi(A) = \text{Tr}(AT)$ with T positive trace-class, it follows that $\rho_x(a(x)) = \text{Tr}(a(x) T_x)$.
(ii) Let $\gamma \in \mathfrak{S}(\tilde{\mathcal{L}})$ be as above in (i), i.e.

$$\gamma(a) := \int_X \text{Tr}(a(x) T_x) d\nu(x)$$

where $a \in \tilde{\mathcal{L}}$. Now we know the unique extension of γ to \mathcal{C}' is given by

$$\tilde{\gamma}(a) = \lim_{\alpha} \gamma(E_{\alpha} a), \quad a \in \mathcal{C}'.$$

In particular, for the states $\gamma_x \in \mathfrak{S}(\tilde{\mathcal{L}})$ given by $\gamma_x(a) := \text{Tr}(a(x) T_x)$, we get

$$\tilde{\gamma}_x(a) = \lim_{\alpha} \text{Tr}(E_{\alpha}(x) a(x) T_x) = \text{Tr}(a(x) T_x)$$

since γ_x is normal, so has a unique extension by the last formula to $\mathcal{B}(\mathcal{H}_{\gamma_x})$. Furthermore, $\|\tilde{\gamma}_x\| = 1$ for all x , and as ν is a probability measure on X , the function 1 is in $L^1(X, \nu)$. Thus we may apply the Lebesgue dominated convergence theorem:

$$\begin{aligned} \tilde{\gamma}(a) &= \lim_{\alpha} \int_X \text{Tr}(E_{\alpha}(x) a(x) T_x) d\nu(x) \\ &= \int_X \lim_{\alpha} \text{Tr}(E_{\alpha}(x) a(x) T_x) d\nu(x) \\ &= \int_X \text{Tr}(a(x) T_x) d\nu(x). \end{aligned}$$

■

Theorem 2.10. γ is a normal state of \mathcal{C}' or of $\pi_{\omega}(\mathcal{F})'' \subset \mathcal{C}'$ iff it can be written

$$\gamma(A) = \int_X \text{Tr}(\pi_{\psi_x}(A) T_x) \cdot f(x) d\mu(x), \quad A \in \mathcal{C}'$$

where $T_x \in \mathcal{K}(\mathcal{H}_{\psi_x})$ is a.e. trace-class, positive and normalised, and $f \in L^1_+(X, \mu)$ with μ the measure associated to the initial choice ω and \mathcal{C} .

Proof: Let γ be a normal state on \mathcal{C}' or $\pi_\omega(\mathcal{F})''$, then by theorem 2.2 there is a unitary $U : \mathcal{H}_\omega \rightarrow \mathcal{H}_\Gamma$ intertwining π_ω with Π . Thus by Kadison and Ringrose 7.1.12 [KR], there is a countable set of vectors $\{\varphi_n\} \subset \mathcal{H}_\Gamma$ such that $1 = \sum_{n=1}^{\infty} \|\varphi_n\|^2$ and $\gamma(A) = \sum_n (\varphi_n, \Pi(A)\varphi_n)$. Now let $\{\zeta_k^n\} \subset \Pi(\mathcal{C}')\Omega$ be sequences such that $\kappa(\zeta_k^n) \rightarrow \varphi_n \in \mathcal{H}_\Gamma$ where the convergence is in k . We can in fact choose such sequences with $\|\kappa(\zeta_k^n)\| = \|\varphi_n\|$ because if ζ_k^n is a nonzero sequence converging to φ_n , so is $\zeta_k^n \|\varphi_n\| / \|\zeta_k^n\|$. Below we will blur the distinction between $\kappa(\zeta_k^n)$ and ζ_k^n . Thus

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} \|\varphi_n\|^2 = \sum_n \|\zeta_k^n\|^2 = \sum_n \int_X \|\zeta_k^n(x)\|_{\mathcal{H}_{\psi_x}}^2 d\mu(x) \\ &= \int_X \sum_n \|\zeta_k^n(x)\|_{\mathcal{H}_{\psi_x}}^2 d\mu(x) \end{aligned}$$

by Fubini and absolute convergence. Thus

$$\begin{aligned} \sum_n \|\zeta_k^n(x)\|_{\mathcal{H}_{\psi_x}}^2 &\in L^1(X, \mu)_+. \quad \text{Now} \\ \left| (\zeta_k^n(x), B \zeta_k^n(x))_{\mathcal{H}_{\psi_x}} \right| &\leq \|B\| \cdot \|\zeta_k^n(x)\|_{\mathcal{H}_{\psi_x}}^2 \quad \text{hence} \\ \gamma_x^k(B) &:= \sum_n (\zeta_k^n(x), B \zeta_k^n(x))_{\mathcal{H}_{\psi_x}} \end{aligned}$$

defines a positive functional on $\mathcal{B}(\mathcal{H}_{\psi_x})$ for μ -almost all x . Moreover, it is normal by Kadison and Ringrose 7.1.12 [KR]. Now for A a positive element of \mathcal{C}'

$$\gamma(A) = \sum_n (\varphi_n, A\varphi_n) = \sum_n \lim_k (\zeta_k^n, A\zeta_k^n) = \lim_k \sum_n (\zeta_k^n, A\zeta_k^n)$$

by dominated convergence, since $|(\zeta_k^n, A\zeta_k^n)| \leq \|A\| \cdot \|\zeta_k^n\|^2 = \|A\| \cdot \|\varphi_n\|^2$. So

$$\begin{aligned} \gamma(A) &= \lim_k \sum_n \int_X (\zeta_k^n(x), A(x) \zeta_k^n(x))_{\mathcal{H}_{\psi_x}} d\mu(x) \\ &= \lim_k \int_X \sum_n (\zeta_k^n(x), A(x) \zeta_k^n(x))_{\mathcal{H}_{\psi_x}} d\mu(x) \\ &= \lim_k \int_X \gamma_x^k(A(x)) d\mu(x) \end{aligned}$$

where we used Fubini's theorem and absolute convergence. Moreover, since

$$|\gamma_x^k(A(x))| \leq \|A\| \sum_n \|\zeta_k^n(x)\|_{\mathcal{H}_{\psi_x}}^2 \in L^1(X, \mu)_+$$

we can use the dominated convergence theorem to conclude

$$\gamma(A) = \int_X \lim_k \gamma_x^k(A(x)) d\mu(x) \quad (1)$$

providing we can show that the pointwise limits $\lim_k \gamma_x^k(A(x))$ exist a.e. which is what we prove now. Since $\zeta_k^n \in \Pi(\mathcal{C}')\Omega$, let $\zeta_k^n = \Pi(A_k^n)\Omega$, $A_k^n \in \mathcal{C}'$, so for $B \in \mathcal{C}'$

$$\begin{aligned} \gamma_x^k(B) &= \sum_n (\zeta_k^n(x), B(x) \zeta_k^n(x)) = \sum_n (\Pi(A_k^n)\Omega(x), \Pi(BA_k^n)\Omega(x)) \\ &= \sum_n \psi_x(A_k^{n*} B A_k^n) \quad \text{so} \\ |\gamma_x^k(B) - \gamma_x^\ell(B)| &= \left| \sum_n \psi_x(A_k^{n*} B A_k^n - A_\ell^{n*} B A_\ell^n) \right| \\ &\leq \left| \sum_n (\psi_x((A_k^n - A_\ell^n)^* B (A_k^n - A_\ell^n)) + \psi_x(A_\ell^{n*} B (A_k^n - A_\ell^n)) \right. \\ &\quad \left. + \psi_x((A_k^n - A_\ell^n)^* B A_\ell^n)) \right| \\ &\leq \|B\| \sum_n (\psi_x((A_k^n - A_\ell^n)^* (A_k^n - A_\ell^n)) \\ &\quad + 2\psi_x(A_\ell^{n*} A_\ell^n)^{1/2} \cdot \psi_x((A_k^n - A_\ell^n)^* (A_k^n - A_\ell^n))^{1/2}) \end{aligned} \quad (2)$$

using the Cauchy–Schwartz inequality. Now

$$\sum_n \psi_x((A_k^n - A_\ell^n)^* (A_k^n - A_\ell^n)) = \|\zeta_k^n(x) - \zeta_\ell^n(x)\|^2$$

which must converge to zero a.e. as k and ℓ approach infinity because

$$\|\zeta_k^n - \zeta_\ell^n\| = \left(\int_X \|\zeta_k^n(x) - \zeta_\ell^n(x)\|^2 d\mu(x) \right)^{1/2}$$

and this approaches zero with k, ℓ since $\{\zeta_k^n\}$ is a convergent sequence. Applying the Cauchy–Schwartz inequality to the sum:

$$\begin{aligned} &\left| \sum_n \psi_x(A_\ell^{n*} A_\ell^n)^{1/2} \cdot \psi_x((A_k^n - A_\ell^n)^* (A_k^n - A_\ell^n))^{1/2} \right|^2 \\ &\leq \sum_n \psi_x(A_\ell^{n*} A_\ell^n) \cdot \sum_m \psi_x((A_k^m - A_\ell^m)^* (A_k^m - A_\ell^m)) \end{aligned}$$

which as we saw approach zero a.e. Thus (2) converges to zero a.e. and so the pointwise limit $\lim_k \gamma_x^k(B(x))$ exists a.e. and (1) is justified.

Since the normal states are sequentially weak*–closed (cf. Takesaki 5.2 p148

[Tak]), we conclude that $\lim_k \gamma_x^k$ is a normal state on $\mathcal{B}(\mathcal{H}_{\psi_x})$. Thus there is a positive normalised trace-class operator $T_x \in \mathcal{K}(\mathcal{H}_{\psi_x})$ such that $\gamma_x(A(x)) = f(x) \cdot \text{Tr}(A(x) T_x)$ where $f(x) = \gamma_x(\mathbb{I}) > 0$. That $f \in L^1(X, \mu)$ follows from

$$1 = \gamma(\mathbb{I}) = \int_X \gamma_x(\mathbb{I}) d\mu(x) = \int_X f(x) d\mu(x).$$

$$\text{Thus } \gamma(A) = \int_X \text{Tr}(\pi_{\psi_x}(A) T_x) f(x) d\mu(x) \quad \forall A \in \mathcal{C}'.$$

Conversely, assume $\gamma(A)$ to have this form. Then by Kadison and Ringrose 7.1.12 [KR], it suffices to show γ is strong operator continuous on the unit ball of \mathcal{C}' . Let $\{A_\alpha\} \subset \mathcal{C}'$ be a net converging to zero in strong operator topology and with $\|A_\alpha\| \leq 1$. That is, for all $\varphi \in \mathcal{H}_\omega$

$$\|A_\alpha \varphi\|^2 = \int_X \|A_\alpha(x) \varphi(x)\|_{\mathcal{H}_{\psi_x}}^2 d\mu(x) \longrightarrow 0$$

which implies that $\|A_\alpha(x) \varphi(x)\| \rightarrow 0$ almost everywhere. Since $A_\alpha(x) = \pi_{\psi_x}(A_\alpha)$,

$$|\gamma(A_\alpha)| = \left| \int_X \text{Tr}(\pi_{\psi_x}(A_\alpha) T_x) f(x) d\mu(x) \right| \leq \int_X |\text{Tr}(\pi_{\psi_x}(A_\alpha) T_x)| f(x) d\mu(x).$$

Moreover $|\text{Tr}(\pi_{\psi_x}(A_\alpha) T_x)| f(x) \leq \|A_\alpha\| \cdot f(x)$ which is of course an L^1 -function and $\lim_\alpha |\text{Tr}(\pi_{\psi_x}(A_\alpha) T_x)| \rightarrow 0$ because $\|\pi_{\psi_x}(A_\alpha)\| \leq 1$ and $\text{Tr}(\cdot T_x)$ is a normal functional on $\mathcal{B}(\mathcal{H}_{\psi_x})$ (recall that $A_\alpha(x)$ converges almost everywhere to 0 in the strong operator topology). Thus by the dominated convergence theorem,

$$\begin{aligned} \lim_\alpha |\gamma(A_\alpha)| &\leq \lim_\alpha \int_X |\text{Tr}(\pi_{\psi_x}(A_\alpha) T_x)| f(x) d\mu(x) \\ &= \int_X \lim_\alpha |\text{Tr}(\pi_{\psi_x}(A_\alpha) T_x)| f(x) d\mu(x) = 0. \end{aligned}$$

The argument generalises to nets $A_\alpha \rightarrow A \neq 0$ in strong operator topology for $\|A_\alpha\| \leq 1$, hence γ is normal. ■

Remark. Observe that the theorem automatically constructs a state on \mathcal{C}' even if one starts from a state on $\pi_\omega(\mathcal{F})''$, though due to the choices involved this need not be unique.

Def. Given \mathcal{F} , ω , \mathcal{C} as above, define the set $\mathfrak{S}_\omega \subset \mathfrak{S}(\tilde{\mathcal{L}})$ as the set of those states γ with $\gamma(a) = \int_X \text{Tr}(a(x) T_x) d\nu(x)$ as in 2.9, where ν is absolutely continuous with respect to μ . As before, we denote the set of normal states of a concrete von Neumann algebra \mathcal{N} by $\mathfrak{S}_N(\mathcal{N})$.

Theorem 2.11. Assume the data and notation of Corr. 2.9, then

(i) For a state $\gamma \in \mathfrak{S}(\tilde{\mathcal{L}})$ the extension $\theta(\gamma)$ is normal on \mathcal{C}' or

on $\pi_\omega(\mathcal{F})''$ iff its measure ν on X is absolutely continuous with the measure μ associated with ω .

(ii) For each normal state η of \mathcal{C}' or $\pi_\omega(\mathcal{F})''$ there is a $\gamma \in \mathfrak{S}(\tilde{\mathcal{L}})$ such that $\eta = \theta(\gamma)$.

(iii) We have: $\theta : \mathfrak{S}_\omega \rightarrow \mathfrak{S}_N(\pi_\omega(\mathcal{F})'')$ is a surjection. Since $\tilde{\mathcal{L}} \subset \mathcal{C}'$, we see that $\theta : \mathfrak{S}_\omega \rightarrow \mathfrak{S}_N(\mathcal{C}')$ is a bijection.

Proof: (i) By Corr. 2.9(ii) and theorem 2.10, this follows immediately.

(ii) Now $\mathcal{C}' \subset \tilde{\mathcal{L}}''$, so normal states of \mathcal{C}' are restrictions of normal states of $\tilde{\mathcal{L}}''$, and these in turn are the unique extensions of states from $\tilde{\mathcal{L}}$ by strong operator continuity. Since $\mathcal{C}' \subset M(\tilde{\mathcal{L}}) \subset \tilde{\mathcal{L}}''$, these strong operator extensions are just the unique extensions by θ . Thus we have surjectivity as claimed.

(iii) This is just a restatement of the preceding parts. ■

Remark. Since \mathfrak{S}_ω is a proper subset of $\mathfrak{S}(\tilde{\mathcal{L}})$, this means that $\tilde{\mathcal{L}}$ cannot in general be an ideal host for \mathcal{C}' or for $\pi_\omega(\mathcal{F})''$, a fact which is also obvious from the commutative situation: $\pi_\omega(\mathcal{F})'' = L^\infty(X, \mu) = \tilde{\mathcal{L}}$. But we have almost finished showing that it is a quasi-host.

Theorem 2.12. Let $\mathcal{C} \cong L^\infty(X, \mu)$ be maximally commutative in $\pi_\omega(\mathcal{F})'$. Then $\mathcal{C}' = (\mathcal{N} \cup \mathcal{C})'' \supset \tilde{\mathcal{L}}$, $\pi_\omega(\mathcal{F}) \subset M(\tilde{\mathcal{L}})$,

$$\mathfrak{S}_\omega = \left\{ \varphi \in \mathfrak{S}(\tilde{\mathcal{L}}) \mid \tilde{\varphi} \upharpoonright L^\infty(X, \mu) \text{ is normal} \right\},$$

$\tilde{\mathfrak{S}}_\omega \upharpoonright \pi_\omega(\mathcal{F}) = \mathfrak{S}_N(\mathcal{N}) = \mathfrak{S}_N(\pi_\omega(\mathcal{F}))$ and $\tilde{\mathfrak{S}}_\omega \upharpoonright C^*(\mathcal{N} \cup \mathcal{C})$ is in bijection with \mathfrak{S}_ω . In short, $\tilde{\mathcal{L}}$ is a quasi-host for $(\mathcal{N}, \mathfrak{S}_N(\mathcal{N}))$ (hence for $(\pi_\omega(\mathcal{F}), \mathfrak{S}_N(\pi_\omega(\mathcal{F})))$).

Proof: Since \mathcal{C} is maximally commutative in \mathcal{N}' , we have $\mathcal{C} = \mathcal{N}' \cap \mathcal{C}' = (\mathcal{N} \cup \mathcal{C})'$ and thus $\mathcal{C}' = (\mathcal{N} \cup \mathcal{C})''$ hence a normal state on \mathcal{C}' is uniquely determined by its values on \mathcal{C} and on \mathcal{N} . Thus by Theorem 2.11(iii) we have that $\tilde{\mathfrak{S}}_\omega \upharpoonright C^*(\mathcal{N} \cup \mathcal{C})$ is in bijection with \mathfrak{S}_ω . We already have the embeddings stated, so to check the claimed characterisation of \mathfrak{S}_ω , recall that it consists of states $\gamma \in \mathfrak{S}(\tilde{\mathcal{L}})$ such that

$$\gamma(A) = \int_X \text{Tr}(A(x)T_x) d\nu(x), \quad A \in \tilde{\mathcal{L}}$$

where ν is absolutely continuous w.r.t. μ . Then for $f \in \mathcal{C} = L^\infty(X, \mu)$ we have for any approximate identity $\{E_\alpha\}$ of $\tilde{\mathcal{L}}$:

$$\tilde{\gamma}(f) = \lim_\alpha \gamma(f \cdot E_\alpha) = \lim_\alpha \int_X f(x) \text{Tr}(E_\alpha(x)T_x) d\nu(x) = \int_X f(x) d\nu(x)$$

using the argument in the proof of Corollary 2.9(ii). These are precisely the normal states of $L^\infty(X, \mu)$. This completes the proof. ■

Return now to the original pair $(\mathcal{F}, \mathfrak{S}_0)$ at the start of the investigation, which was equivalent to the examination of $(\pi_{\mathfrak{S}_0}(\mathcal{F}), \mathfrak{S}_N(\pi_{\mathfrak{S}_0}(\mathcal{F})))$, where $\pi_{\mathfrak{S}_0} = \bigoplus_{\omega \in \mathfrak{S}_0} \pi_\omega$. Take the direct sum $\tilde{\mathcal{L}}_0 := \bigoplus_{\omega \in \mathfrak{S}_0} \tilde{\mathcal{L}}_\omega \subset \mathcal{B}(\mathcal{H}_{\mathfrak{S}_0})$ where $\tilde{\mathcal{L}}_\omega$ is the quasi-host constructed above for the pair $(\pi_\omega(\mathcal{F}), \mathfrak{S}_N(\pi_\omega(\mathcal{F})))$. Then

$$\pi_{\mathfrak{S}_0}(\mathcal{F}) \subset M(\tilde{\mathcal{L}}_0), \quad \text{and} \quad \bigoplus_{\omega \in \mathfrak{S}_0} L^\infty(X_\omega, \mu_\omega) \subset ZM(\tilde{\mathcal{L}}_0).$$

Let (X, μ) be the disjoint union of the measure spaces (X_ω, μ_ω) (hence μ is not a probability measure), so we can write $L^\infty(X, \mu) = \bigoplus_{\omega \in \mathfrak{S}_0} L^\infty(X_\omega, \mu_\omega)$, and thus $L^\infty(X, \mu) \subset ZM(\tilde{\mathcal{L}}_0)$. Moreover with notation

$$\mathfrak{S}_\mu := \left\{ \varphi \in \mathfrak{S}(\tilde{\mathcal{L}}_0) \mid \tilde{\varphi} \upharpoonright L^\infty(X, \mu) \text{ is normal} \right\}$$

we see that for $\varphi \in \mathfrak{S}_\mu$ that $\tilde{\varphi} \upharpoonright L^\infty(X_\omega, \mu_\omega)$ is normal for all $\omega \in \mathfrak{S}_0$, hence \mathfrak{S}_μ is the norm closed convex hull of $\bigcup_{\omega \in \mathfrak{S}_0} \mathfrak{S}_\omega$ and by Theorem 2.12 $\tilde{\mathfrak{S}}_\mu \upharpoonright \pi_{\mathfrak{S}_0}(\mathcal{F}) = \mathfrak{S}_0$ and $\tilde{\mathfrak{S}}_\mu \upharpoonright C^*(\pi_{\mathfrak{S}_0}(\mathcal{F}) \cup L^\infty(X, \mu))$ is in bijection with \mathfrak{S}_μ . In other words, $\tilde{\mathcal{L}}_0$ is a quasi-host for $(\mathcal{F}, \mathfrak{S}_0)$. Note that whilst $\tilde{\mathcal{L}}_0$ is in $\mathcal{B}(\mathcal{H}_{\mathfrak{S}_0})$, it need not be in $\pi_{\mathfrak{S}_0}(\mathcal{F})''$, (unlike ideal hosts) because \mathcal{C} may have a part outside $\pi_{\mathfrak{S}_0}(\mathcal{F})''$.

Theorem 2.13. *Given the preceding notation, if $(\mathcal{F}, \mathfrak{S}_0)$ has an ideal host, then μ must have discrete points.*

Proof: If $(\mathcal{F}, \mathfrak{S}_0)$ has an ideal host, then by Corollary 1.7, \mathfrak{S}_0 has pure states. So in the direct sum $\pi_{\mathfrak{S}_0}(\mathcal{F}) = \bigoplus_{\omega \in \mathfrak{S}_0} \pi_\omega(\mathcal{F})$ we have that for some $\omega \in \mathfrak{S}_0$, that $\pi_\omega(\mathcal{F})' = \mathbb{C}\mathbb{I}$, and so $\mathcal{C}_\omega = \mathbb{C}\mathbb{I}$ and thus (X_ω, μ_ω) is the discrete trivial measure space $(\{x\}, \delta)$, $\delta(\{x\}) = 1$, $\delta(\emptyset) = 0$. Since this is a summand of $L^\infty(X, \mu)$, we conclude that μ has discrete points. ■

This makes now explicit the sense in which we meant that $L^\infty(X, \mu)$ with μ continuous, is an obstruction to the existence of an ideal host. It would be nice to get a converse, i.e. to argue that an ideal host exists iff μ has some specific structure, but we do not have this yet.

3. Applications.

Here we will do a couple of applications of ideal hosts, mainly following the well-trodden path of group algebras. First, there is the question of useful decompositions of states in \mathfrak{S}_0 into other states in \mathfrak{S}_0 :

Theorem 3.1. *Let \mathcal{L} be an ideal host for a pair $(\mathcal{F}, \mathfrak{S}_0)$, and let $\varphi \in \mathfrak{S}_0$. Then there is an integral decomposition*

$$\varphi(F) = \int_{\mathfrak{S}_0} \omega(F) d\mu(\omega), \quad F \in \mathcal{F},$$

such that μ is pseudosupported on the pure states of \mathfrak{S}_0 . If \mathcal{L} is separable, μ is actually supported on the pure states of \mathfrak{S}_0 .

Proof: Most of the work for the proof has already been done in the last section. First observe that from Theorem 1.5 we know that $\mathcal{L}'' = \pi_{\mathfrak{S}_0}(\mathcal{F})''$, and hence for an $\omega \in \mathfrak{S}(\mathcal{L})$ we have that $\pi_\omega(\mathcal{L})'' = \pi_{\theta(\omega)}(\mathcal{F})''$. For a given $\varphi \in \mathfrak{S}_0$, choose $\omega = \theta^{-1}(\varphi)$. From Theorem 2.1, for a choice of commutative algebra $\mathcal{C} \subset \pi_\omega(\mathcal{L})'$, we have the decomposition

$$\tilde{\omega}(A) := (\Omega_\omega, A\Omega_\omega) = \int_X \psi_x(A) d\mu(x)$$

for all $A \in \mathcal{C}' \supset \pi_\omega(\mathcal{F}) = \pi_\varphi(\mathcal{F})$, and hence by restriction to $\pi_\varphi(\mathcal{F})$ this is also a decomposition for $\varphi = \tilde{\omega} \upharpoonright \mathcal{F} \in \mathfrak{S}_0$. Make the usual identification of X with a subset of $\mathfrak{S}(\mathcal{L})$ by $x \rightarrow \psi_x$, and maintain the same symbol for the measure μ carried by this identification to $\mathfrak{S}(\mathcal{L})$. Of course via θ we can now carry this measure to \mathfrak{S}_0 too. Thus we have

$$\begin{aligned} \omega(L) &= \int_{\mathfrak{S}(\mathcal{L})} \gamma(L) d\mu(\gamma), \quad L \in \mathcal{L}, \quad \text{hence:} \\ \varphi(F) &= \theta(\omega)(F) = \lim_{\alpha} \int_{\mathfrak{S}(\mathcal{L})} \gamma(F \cdot E_\alpha) d\mu(\gamma) \\ &= \int_{\mathfrak{S}(\mathcal{L})} \theta(\gamma)(F) d\mu(\gamma) = \int_{\theta(\mathfrak{S}(\mathcal{L}))} \psi(F) d\mu(\psi) \\ &= \int_{\mathfrak{S}_0} \psi(F) d\mu(\psi) \end{aligned}$$

for all $F \in \mathcal{F}$, any approximate identity $\{E_\alpha\}$ of \mathcal{L} , and where we used the argument in the proof of Corollary 2.9(ii) to bring the limit into the integral. This then provides the basic decomposition formula. Now, since the measure μ on $\mathfrak{S}(\mathcal{L})$ is the same one constructed in the decomposition theory in Takesaki [Tak], we can use his Theorem 6.28, p246, to conclude that if \mathcal{C} is maximally commutative in $\pi_\omega(\mathcal{L})'$, then μ is pseudosupported by the pure states of \mathcal{L} , and supported by them if \mathcal{L} is separable. Because θ restricts to a bijection

between the pure states on \mathcal{L} and the pure states of \mathfrak{S}_0 , this proves the assertion. ■

Remark. Since by Theorem 1.5 we deduce that θ also restricts to a bijection between the factor states of \mathcal{L} and the factor states of \mathfrak{S}_0 , we can use the central decomposition of a state on \mathcal{L} to obtain a similar decomposition of a state in \mathfrak{S}_0 , in terms of factor states. Again, we will have the two claims for the measure;– in general it is pseudosupported on the factor states, but if \mathcal{L} is separable, it is actually supported by the factor states.

As a second application for host algebras, we mention that of inducing representations. For group algebras, this is of course one of the main applications (cf. Rieffel [Ri]). What allows one to do the same here, is the relationship given in Theorem 1.9 between the representations of \mathcal{F} and those of a host algebra. To be more specific, given two pairs $(\mathcal{F}_i, \mathfrak{S}_{0i})$, $i = 1, 2$ together with respective hosts \mathcal{L}_i then providing one has constructed a right \mathcal{L}_1 -rigged left \mathcal{L}_2 -module \mathcal{M} , then we can induce a representation $\pi \in \text{Rep}_{\mathfrak{S}_{01}} \mathcal{F}_1$ to a representation $\rho \in \text{Rep}_{\mathfrak{S}_{02}} \mathcal{F}_2$ by using the map Λ in Theorem 1.9 to identify π with a representation of \mathcal{L}_1 , inducing this representation via \mathcal{M} to \mathcal{L}_2 , and then identifying the result with a representation $\rho \in \text{Rep}_{\mathfrak{S}_{02}} \mathcal{F}_2$ with the map Λ . The benefit of doing an induction via host algebras is that we remain within the class of representations normal w.r.t. the representations $\pi_{\mathfrak{S}_{0i}}$ whereas induction between the \mathcal{F}_i 's directly, can move us out of these classes. The crux comes of course with the construction of the module \mathcal{M} . For concrete examples of this, we refer to any example of induction via group algebras (cf. Rieffel [Ri]).

Acknowledgements.

I am deeply grateful to Detlev Buchholz, who has been as much responsible for the development of host algebras as I have. Not only did the current line of thought develop out of his question at a seminar I gave in 1997 and many subsequent lively discussions, but he also read several previous misguided attempts, discovering deeply buried and serious errors. Moreover, over the years he prodded me to return to this project, and most recently at Göttingen provided the opportunity and support for me to pursue it, of which this paper is the result. Thank you Detlev!

I am also grateful to the Erwin Schrödinger Institute in Vienna (and in particular to Jakob Yngvason and Heide Narnhofer) who provided me with the opportunity of completing the last part of the paper.

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